Appendices to "The EPR Paradox, Einstein-Rosen bridges and teleportation"

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A Obtaining the energy-momentum tensor produced by an electromagnetic field

The definition of the energy-momentum tensor is[1] :

$$T_{\mu\nu} = -2\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}, \tag{A.1}$$

where $g_{\mu\nu}$ is the metric and S_M is the action of the field. If we suppose that the field is an electromagnetic field, we have the following Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\sqrt{-g}, \tag{A.2}$$

with A_μ being the electromagnetic potential and J^μ being the electric current. We get:

$$\begin{aligned} S_M &= -\frac{1}{4}\int \sqrt{-g}F_{\mu\nu}F^{\mu\nu}\,\mathrm{d}^4x \\ &= -\frac{1}{4}\int \sqrt{-g}F_{\mu\nu}F_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}\,\mathrm{d}^4x. \end{aligned} \tag{A.3}$$

We can now deduce the energy-momentum tensor of a photon.

Lemma 1. *The energy-momentum tensor produced by a photon is given by*

$$T_{\mu\nu} = F_{\mu\rho}g^{\rho\sigma}F_{\sigma\nu} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} \tag{A.4}$$

Proof. From equation A.3, we obtain the variation of the action:

$$\begin{aligned} \delta S_M &= -\frac{1}{4}\left[\int \mathrm{d}^4x\, F_{\mu\nu}F^{\mu\nu}\,\delta\sqrt{-g} + \int \mathrm{d}^4x\,\sqrt{-g}\,\delta\left(F_{\rho\sigma}F^{\rho\sigma}\right)\right] \\ &= -\frac{1}{4}\left(\delta S_1 + \delta S_2\right). \end{aligned} \tag{A.5}$$

Since

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}, \tag{A.6}$$

we get

$$\delta S_1 = -\frac{1}{2}\int \mathrm{d}^4x\,\sqrt{-g}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\,\delta g^{\mu\nu}. \tag{A.7}$$

For δS_2, note that

$$\frac{\delta\left(F_{\rho\sigma}F^{\rho\sigma}\right)}{\delta g^{\mu\nu}} = 2g^{\rho\sigma}F_{\rho\mu}F_{\sigma\nu}, \tag{A.8}$$

hence:

$$\Rightarrow T_{\mu\nu} = -2\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} = g^{\rho\sigma}F_{\rho\mu}F_{\sigma\nu} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}. \tag{A.9}$$

□

We also have

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu, \tag{A.10}$$

and the homogeneous electromagnetic wave equation[2]

$$-\nabla^\beta\nabla_\beta A^\alpha + R_\beta{}^\alpha A^\alpha = J^\alpha. \tag{A.11}$$

bearing in mind that

$$\nabla_\mu A^\mu = 0. \tag{A.12}$$

B Solving the wave equation

Suppose that we use the Schwarzschild metric

$$g_{\mu\nu} = \begin{pmatrix} -\left(1-\frac{2GM}{r}\right) & 0 & 0 & 0 \\ 0 & \left(1-\frac{2GM}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2(\theta) \end{pmatrix} \tag{B.1}$$

in the coordinate system $(t, r.\theta, \phi)$. The wave equation thus becomes:

$$-g^{\beta\mu}\left[\partial_\mu\partial_\beta A^\alpha + \partial_\mu\Gamma^\alpha_{\beta\lambda}A^\lambda + \Gamma^\alpha_{\beta\lambda}\partial_\mu A^\lambda + \Gamma^\alpha_{\mu\lambda}\left(\partial_\beta A^\lambda + \Gamma^\lambda_{\beta\gamma}A^\gamma\right)\right] + g^{\alpha\nu}R_{\mu\lambda\beta}{}^\lambda A^\beta = 0, \tag{B.2}$$

hence, applying ths metric gives us, using the fact that $\forall\alpha,\beta,\gamma,\ \partial_t\Gamma^\alpha_{\beta\gamma} = \partial_\phi\Gamma^\alpha_{\beta\gamma} = 0$:

$$\begin{cases}
\bullet\ -g^{tt}\left[\partial_t\partial_t A^t + \Gamma^t_{tr}\partial_t A^r + \Gamma^t_{tr}\left(\partial_t A^r + \Gamma^r_{tt}A^t\right)\right] & \\
-g^{rr}\left[\partial_r\partial_r A^t + \partial_r\Gamma^t_{rt}A^t + \Gamma^t_{rt}\partial_r A^t + \Gamma^t_{rt}\left(\partial_r A^t + \Gamma^t_{rt}A^t\right)\right] - g^{\theta\theta}\partial_\theta\partial_\theta A^t - g^{\phi\phi}\partial_\phi\partial_\phi A^t & \\
+g^{tt}R_{tt}A^t & = J^t \\
\bullet\ -g^{tt}\left[\partial_t\partial_t A^r + \Gamma^r_{tt}\partial_t A^t + \Gamma^r_{tt}\left(\partial_t A^t + \Gamma^t_{tr}A^r\right)\right] & \\
-g^{rr}\left[\partial_r\partial_r A^r + \partial_r\Gamma^r_{rr}A^r + \Gamma^r_{rr}\partial_r A^r + \Gamma^r_{rr}\left(\partial_r A^r + \Gamma^r_{rr}A^r\right)\right] & \\
-g^{\theta\theta}\left[\partial_\theta\partial_\theta A^r + \Gamma^r_{\theta\theta}\partial_\theta A^\theta + \Gamma^r_{\theta\theta}\left(\partial_\theta A^\theta + \Gamma^\theta_{\theta r}A^r\right)\right] & \\
-g^{\phi\phi}\left[\partial_\phi\partial_\phi A^r + \Gamma^r_{\phi\phi}\partial_\phi A^\phi + \Gamma^r_{\phi\phi}\left(\partial_\phi A^\phi + \Gamma^\phi_{\phi r}A^r\right)\right] + g^{rr}R_{rr}A^r & = J^r \\
\bullet\ -g^{tt}\partial_t\partial_t A^\theta - g^{rr}\left[\partial_r\partial_r A^\theta + \partial_r\Gamma^\theta_{r\theta}A^\theta + \Gamma^\theta_{r\theta}\partial_r A^\theta + \Gamma^\theta_{r\theta}\left(\partial_r A^\theta + \Gamma^\theta_{r\theta}A^\theta\right)\right] & \\
-g^{\theta\theta}\left[\partial_\theta\partial_\theta A^\theta + \Gamma^\theta_{\theta r}\partial_\theta A^r + \Gamma^\theta_{\theta r}\left(\partial_\theta A^r + \Gamma^r_{\theta\theta}A^\theta\right)\right] & \\
-g^{\phi\phi}\left[\partial_\phi\partial_\phi A^\theta + \Gamma^\theta_{\phi\phi}\partial_\phi A^\theta + \Gamma^\theta_{\phi\phi}\left(\partial_\phi A^\phi + \Gamma^\phi_{\phi\theta}A^\theta\right)\right] + g^{\theta\theta}R_{\theta\theta}A^\theta & = J^\theta \\
\bullet\ -g^{tt}\partial_t\partial_t A^\phi - g^{rr}\left[\partial_r\partial_r A^\phi + \partial_r\Gamma^\phi_{r\phi}A^\phi + \Gamma^\phi_{r\phi}\partial_r A^\phi + \Gamma^\phi_{r\phi}\left(\partial_r A^\phi + \Gamma^\phi_{r\phi}A^\phi\right)\right] & \\
-g^{\theta\theta}\left[\partial_\theta\partial_\theta A^\phi + \partial_\theta\Gamma^\phi_{\theta\phi}A^\phi + \Gamma^\phi_{\theta\phi}\partial_\theta A^\phi + \Gamma^\phi_{\theta\phi}\left(\partial_\theta A^\phi + \Gamma^\phi_{\theta\phi}A^\phi\right)\right] & \\
-g^{\phi\phi}\left[\partial_\phi\partial_\phi A^\phi + \Gamma^\phi_{\phi r}\partial_\phi A^r + \Gamma^\phi_{\phi\theta}\partial_\phi A^\theta + \Gamma^\phi_{\phi r}\left(\partial_\phi A^r + \Gamma^r_{\phi\phi}A^\phi\right) + \Gamma^\phi_{\phi\theta}\left(\partial_\phi A^\theta + \Gamma^\theta_{\phi\phi}A^\phi\right)\right] & \\
+g^{\phi\phi}R_{\phi\phi}A^\phi & = J^\phi.
\end{cases} \tag{B.3}$$

Suppose that we have $A^\mu = A^\mu(t,r)$, $J^\mu = 0, \forall\mu \neq r$, and that $A^\theta = A^\phi = 0$. Using the fact that

$$\nabla_\mu A^\mu = 0 \Rightarrow \partial_t A^t + \Gamma^t_{tr}A^r + \partial_r A^r + \Gamma^r_{rr}A^r = 0, \tag{B.4}$$

as well as using the values of the Christoffel symbol and Ricci curvature tensor:

$$\begin{array}{ccc} \Gamma^{t}_{tr} = \frac{GM}{r(r-2GM)} & \Gamma^{r}_{tt} = \frac{GM}{r^2}\left(1-\frac{2GM}{r}\right) & \Gamma^{r}_{rr} = -\frac{GM}{r(r-2GM)} \\ \Gamma^{\theta}_{r\theta} = \frac{1}{r} & \Gamma^{r}_{\theta\theta} = -r\left(1-\frac{2GM}{r}\right) & \Gamma^{\phi}_{r\phi} = \frac{1}{r} \\ \Gamma^{r}_{\phi\phi} = -r\left(1-\frac{2GM}{r}\right)\sin^2\theta & \Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta & \Gamma^{\phi}_{\theta\phi} = \frac{\cos\theta}{\sin\theta}, \end{array} \tag{B.5}$$

and

$$\begin{aligned} R_{tt} &= \left(1-\frac{2GM}{r}\right)^2\left[\frac{2GM}{r^2(r-2GM)} + 2\left(\frac{GM}{r(r-2GM)}\right)^2 - \frac{2GM(r-GM)}{r^2(r-2GM)^2}\right] \\ R_{rr} &= -\left(1-\frac{2GM}{r}\right)^{-2} R_{tt}, \end{aligned} \tag{B.6}$$

we can simplify the wave equation to:

$$\begin{cases} \frac{1}{\left(1-\frac{2GM}{r}\right)}\partial_t^2 A^t + \frac{2GM}{(r-2GM)^2}\partial_t A^r - \left(1-\frac{2GM}{r}\right)\partial_r^2 A^t - \frac{2GM}{r^2}\partial_r A^t + \frac{2GM(r-GM)}{r^3(r-2GM)^2}A^t & = 0 \\ \frac{1}{\left(1-\frac{2GM}{r}\right)}\partial_t^2 A^r - \frac{2GM}{r^2}\partial_t A^t - \left(1-\frac{2GM}{r}\right)\partial_r^2 A^r + \frac{GM}{r}\partial_r A^r + \frac{2\left(r^2-5GMr+5G^2M^2\right)}{r^3(r-2GM)}A^r & = J^r \end{cases}. \tag{B.7}$$

Taking now the Fourier decomposition

$$\begin{aligned} A^t &= \int \tilde{A}^t(k,\omega)\, e^{i(kr-\omega t)}\mathrm{d}k\,\mathrm{d}\omega \\ A^r &= \int \tilde{A}^r(k,\omega)\, e^{i(kr-\omega t)}\mathrm{d}k\,\mathrm{d}\omega \\ J^r &= \int \tilde{J}^r(k,\omega)\, e^{i(kr-\omega t)}\mathrm{d}k\,\mathrm{d}\omega, \end{aligned} \tag{B.8}$$

we get, using equation B.7:

$$\begin{cases} \int\left(\left[-\frac{\omega^2}{\left(1-\frac{2GM}{r}\right)} + \left(1-\frac{2GM}{r}\right)k^2 - i\frac{2GMk}{r^2} + \frac{2GM(r-GM)}{r^3(r-2GM)^2}\right]\tilde{A}^t(k,\omega) \right. & \\ \left. -i\omega\frac{2GM}{(r-2GM)^2}\tilde{A}^r(k,\omega)\right)e^{i(kr-\omega t)}\mathrm{d}k\,\mathrm{d}\omega & = 0 \\ \int\left(\left[-\frac{\omega^2}{\left(1-\frac{2GM}{r}\right)} + \left(1-\frac{2GM}{r}\right)k^2 + i\frac{GMk}{r} + \frac{2\left(r^2-5GMr+5G^2M^2\right)}{r^3(r-2GM)}\right]\tilde{A}^r(k,\omega) \right. & \\ \left. +i\omega\frac{2GM}{r^2}\tilde{A}^t(k,\omega) - \tilde{J}^r(k,\omega)\right)e^{i(kr-\omega t)}\mathrm{d}k\,\mathrm{d}\omega & = 0 \end{cases}. \tag{B.9}$$

Setting

$$f_1(k,\omega,r) = -\frac{\omega^2}{\left(1-\frac{2GM}{r}\right)} + \left(1-\frac{2GM}{r}\right)k^2 - i\frac{2GMk}{r^2} + \frac{2GM(r-GM)}{r^3(r-2GM)^2} \tag{B.10}$$

and

$$f_2(k,\omega,r) = -\frac{\omega^2}{\left(1-\frac{2GM}{r}\right)} + \left(1-\frac{2GM}{r}\right)k^2 + i\frac{GMk}{r} + \frac{2\left(r^2-5GMr+5G^2M^2\right)}{r^3(r-2GM)}, \tag{B.11}$$

we finally get:

$$\begin{cases} \tilde{A}^t & = i\omega\frac{2GM}{f_1(r-2GM)^2}\tilde{A}^r \\ \tilde{A}^r & = \frac{1}{f_2+\frac{4G^2M^2\omega^2}{f_1(r-2GM)^2r^2}}\tilde{J}^r \end{cases}, \tag{B.12}$$

hence

$$\begin{cases} A^t &= \int i\omega \frac{2GM}{f_2 f_1 (r-2GM)^2 + \frac{4G^2M^2\omega^2}{r^2}} \tilde{J}^r e^{i(kr-\omega t)} \mathrm{d}k\, \mathrm{d}\omega \\ A^r &= \int \frac{1}{f_2 + \frac{4G^2M^2\omega^2}{f_1 (r-2GM)^2 r^2}} \tilde{J}^r e^{i(kr-\omega t)} \mathrm{d}k\, \mathrm{d}\omega \end{cases} . \tag{B.13}$$

Supplementary References

[1] Carroll S. Spacetime and geometry : an introduction to general relativity. San Francisco: Addison Wesley; 2004.

[2] Misner C. Gravitation. New York: W.H. Freeman and Company; 1973.