Submitted: 12/12/20

Research Article

Department of Physics, McGill University, Montreal QC H3A 2T8, Canada

Keywords

Hochschild cohomology, Frobenius algebras, 3-manifolds, cohomology algebra

Email Correspondence

qiu.s.wang@mail.mcgill.ca

Qiu Shi Wang[1]

# Hochschild Cohomology of the Cohomology Algebra of Closed Orientable Three-Manifolds


## Abstract

Let $\mathbb{F}$ be a field of characteristic other than $2$. We show that the zeroth Hochschild cohomology vector space $HH^0(A)$ of a degree $3$ graded commutative Frobenius $\mathbb{F}$-algebra $A = \oplus_i A^i$, where we will always assume $A^0 \cong \mathbb{F}$, is isomorphic to the direct sum of the degree $0, 2$ and $3$ graded components and the kernel of a certain natural evaluation map $\iota_\mu : A^1 \to \Lambda^2(A^1)$. In particular, this holds for $A = H^*(M;\mathbb{F})$ the cohomology algebra of a closed orientable $3$-manifold $M$.

In Theorem A of [1], Charette proves the vanishing of a certain discriminant $\Delta$ associated to a closed orientable 3-manifold $L$ with vanishing cup product $3$-form. It turns out that if we could show that $HH^{2,-2}(A) = 0$ for $A = H^*(L;\mathbb{C})$, we would have found a more elementary proof of this part of Charette's theorem. We show that for any $\beta \geq 3$, the degree $3$ graded commutative Frobenius algebra $A$ with $\mu_A = 0$ and $\dim(A^1) = \beta$ satisfies $HH^{2,-2}(A) \neq 0$. Thus Charette's theorem is not simplified.


## 1 Introduction

### Overview of the problems

We first introduce the problem without defining the mathematical objects involved. The definitions will be provided in further sections of the text, with references.

To any $\mathbb{F}$-algebra $A$ we can associate the Hochschild cohomology $HH^*(A;A)$. If $A$ is a graded algebra, then it also has a bigraded version $HH^{*,*}(A;A)$ defined in (2.4). In this text we will always use coefficients in $A$, so we omit it and write $HH^*(A)$ and $HH^{*,*}(A)$.

The cohomology algebra of a closed orientable 3-manifold is a degree 3 graded commutative Frobenius algebra with zeroth graded component of dimension 1. As we are interested in characterizing the cohomology algebra of such manifolds, we restrict our study to the above-mentioned type of algebra.

We are interested in answering the following two questions:

- **Question 1:** Let $A$ be a degree 3 graded commutative Frobenius algebra with $A^0 \cong \mathbb{F}$. Can we compute $HH^*(A)$ and $HH^{*,*}(A)$ in terms of its 3-form $\mu_A$ and $\dim(A^2)$?
- **Question 2:** In particular, if $\mu_A = 0$, is $HH^{2,-2}(A)$ necessarily zero?

### Main results

We give a limited partial answer to Question 1. As the computational complexity of calculating $HH^n(A)$ is of $O(e^n)$, brute force calculation using a computer is infeasible. However, we can still characterize $HH^0(A)$. Define the 3-form of $A$ to be the map $\mu_A : A^1 \times A^1 \times A^1 \to \mathbb{F}$ by $\mu_A(x,y,z) = \sigma(xy,z)$ where $\sigma$ is the Frobenius form of $A$. Define the map $\iota_\mu : A^1 \to \Lambda^2(A^1)$ by $\iota_\mu(x) = \mu_A(x,\cdot,\cdot)$. Then our first main result is:

**Theorem 1.** *Let $\mathbb{F}$ be a field of characteristic other than 2 and let $A$ be a graded commutative Frobenius $\mathbb{F}$-algebra of degree 3, with graded components $A^0, A^1, A^2$ and $A^3$ such that $A^0 \cong \mathbb{F}$. Then*

$$HH^0(A) \cong A^0 \oplus \ker(\iota_\mu) \oplus A^2 \oplus A^3 \cong \mathbb{F}^{2+\dim A^2} \oplus \ker(\iota_\mu).$$

Note that $\dim(A^1) = \dim(A^2)$. Our second main result answers Question 2 in the negative.

**Theorem 2.** *Let $\mathbb{F}$ be a field of characteristic other than 2 and let $\beta \geq 3$ be an integer. Then the unique degree 3 graded commutative Frobenius algebra $A = \oplus_i A^i$ with $A^0 \cong \mathbb{F}$ such that $\dim_{\mathbb{F}}(A^1) = \beta$ and $\mu_A = 0$ satisfies*

$$HH^{2,-2}(A) \neq 0.$$

### Motivation and significance

Recall that for any closed orientable 3-manifold $M$, we have $H^3(M;\mathbb{F}) \cong \mathbb{F}$ by Poincaré duality. We can define an antisymmetric 3-form using the cup product:

$$\mu_M : H^1(M;\mathbb{F}) \times H^1(M;\mathbb{F}) \times H^1(M;\mathbb{F}) \to H^3(M;\mathbb{F}) \cong \mathbb{F}.$$

Together with Poincaré duality, $\mu_M$ uniquely determines the cohomology algebra $H^*(M;\mathbb{F}) \equiv A$, a degree 3 graded commutative Frobenius algebra with 3-form $\mu_A = \mu_M$. Thus, Theorem 1 provides a characterization of the zeroth Hochschild cohomology of the cohomology $\mathbb{F}$-algebra of a closed orientable 3-manifold in terms of its cup product 3-form $\mu_M$.

Another motivation for our work on degree 3 graded commutative Frobenius algebras is the following result due to Sullivan:

**Theorem 3** (Sullivan [2])**.** *Let $\mu$ be an integral antisymmetric 3-form on a free abelian group $H$ of finite rank. Then there exists a closed orientable 3-manifold $M$ such that $\mu$ is the cup product 3-form of $M$ and $H^1(M;\mathbb{Z}) = H$.*

However, Sullivan's theorem is not necessarily true for an arbitrary 3-form on a finite-dimensional $\mathbb{F}$-vector space, and thus not every Frobenius algebra we consider may be realized by a manifold.

In Theorem A of [1], Charette proves the vanishing of a discriminant $\Delta$ associated to a closed orientable Lagrangian 3-manifold $L$ with vanishing cup product 3-form $\mu_L$ by using holomorphic curve techniques. It turns out that $\Delta$ is the discriminant of a quadratic form in the image of a map $\Theta : HH^{2,-2}(H^*(L;\mathbb{C})) \to Q^2(H^1(L;\mathbb{C});\mathbb{C})$ from $HH^{2,-2}$ to the space of complex valued quadratic forms on the first cohomology group of $L$. Then, one can ask if there is a more elementary proof that $\Delta$ vanishes, for example by showing that $HH^{2,-2} = 0$ for any $L$ with $\mu_L = 0$. This is precisely Question 1, to which Theorem 2 answers in the negative. Thus Charette's proof is not simplified. More details can be found in section 2.5.

## 2 Background

### 2.1 Graded commutative Frobenius algebras

Let $\mathbb{F}$ be a field. We recall the following definitions.

An $\mathbb{F}$-algebra $A$ is said to be graded if it can be decomposed into a direct sum $A = \oplus_{n=0}^{\infty} A^n$ such that $A^pA^q \subset A^{p+q}$. The highest $n$ for which $A^n \neq 0$, if it exists, is the degree of the graded algebra. An algebra is said to be graded commutative if it is graded and also, for $x_p \in A^p$ and $x_q \in A^q$, we have

$$x_px_q = (-1)^{pq}x_qx_p. \tag{2.1}$$

We define a graded commutative Frobenius algebra as an associative finite-dimensional unital graded commutative algebra $A = \oplus_{i=0}^{n} A^i$ equipped with a nondegenerate bilinear form $\sigma : A \times A \to \mathbb{F}$ satisfying $\sigma(xy, z) = \sigma(x, yz)$ for all $x, y, z \in A$. We require $\sigma$, the Frobenius form of $A$, to be consistent with the grading of $A$ in the sense that $\sigma|_{A^i \times A^j}$ is the zero map whenever $i + j \neq n$. Note that the unit of the algebra is in $A^0$.

Throughout this article we will only consider degree $n$ graded commutative Frobenius algebras $A = \oplus_{p=0}^{n} A^p$ such that $A^0 \cong \mathbb{F}$. For $n = 3$, we define the 3-form of $A$ to be the map $\mu_A : A^1 \times A^1 \times A^1 \to \mathbb{F}$ sending $(x, y, z)$ to $\sigma(xy, z)$. The proof of a version of the following useful proposition can be found in [3, Section 10.2].

**Proposition 1.** *Given a basis $\{x_1, \cdots, x_b\}$ for $A^p$, there is a basis $\{\overline{x_1}, \cdots, \overline{x_b}\}$ for $A^{n-p}$ dual to it in the sense that $\sigma(x_i, \overline{x_j}) = \delta_{ij}$.*

### 2.2 Cohomology algebra of closed orientable 3-manifolds

The following results from elementary homology theory can be found in any introductory textbook in algebraic topology, notably [4] and [5]. They show that Theorems 1 and 2 apply to cohomology algebras of closed orientable 3-manifolds.

For an abelian coefficient group $G$, the singular cohomology functors $H^i$ : **Top** $\to$ **Ab** take a topological space $X$ to its cohomology groups $H^i(X; G)$. By Poincaré duality, we know that for a closed orientable 3-manifold $M$ and $\mathbb{F}$ a coefficients field, $H^0(M;\mathbb{F}) \cong H^3(M;\mathbb{F}) \cong \mathbb{F}$ and $H^1(M;\mathbb{F}) \cong H^2(M;\mathbb{F}) \cong \mathbb{F}^\beta$, where $\beta$ is the first Betti number of $M$. We have $H^i(M;\mathbb{F}) = 0$ for $i < 0$ or $i \geq 4$. The vector space $H^*(M;\mathbb{F}) = \bigoplus_i H^i(M;\mathbb{F})$, together with the cup product $\smile : H^i(M;\mathbb{F}) \times H^j(M;\mathbb{F}) \to H^{i+j}(M;\mathbb{F})$, forms the cohomology algebra of $M$ with coefficients in $\mathbb{F}$. The cup product is graded commutative, that is, for $x_p \in H^p(M;\mathbb{F})$ and $x_q \in H^q(M;\mathbb{F})$, it satisfies (2.1).

Choose an orientation for $M$ and let $[M] \in H_n(M;\mathbb{F})$ be the corresponding fundamental class. We will need the following consequence of Poincaré duality:

**Theorem 4.** *For a field $\mathbb{F}$ and $M^n$ a closed and orientable manifold, the map*

$$\varphi : H^p(M;\mathbb{F}) \to \mathrm{Hom}_{\mathbb{F}}(H^{n-p}(M;\mathbb{F}), \mathbb{F})$$

*taking $\alpha \mapsto \overline{\alpha}$ where $\overline{\alpha}(x) = (\alpha \smile x)([M])$, is an isomorphism. Equivalently, there is a nondegenerate pairing*

$$H^p(M;\mathbb{F}) \times H^{n-p}(M;\mathbb{F}) \xrightarrow{\langle\,,\,\rangle} \mathbb{F}$$

*sending $(a, b) \mapsto \varphi(a)(b) = (a \smile b)([M])$. Therefore, the algebra $H^*(M;\mathbb{F})$ is a degree $n$ graded commutative Frobenius algebra with Frobenius form $\sigma(a, b) = \langle a, b\rangle$.*

Now let $n = 3$. If we choose the basis $\{e\}$ of $H^3(M;\mathbb{F})$ such that $e([M]) = 1 \in \mathbb{F}$, we get that $x_i \smile \overline{x_j} = \delta_{ij}e$ for $x_i$ and $\overline{x_j}$ from Proposition 1. The latter Proposition notably implies that any nonzero element $x \in A^1$ has a dual $x^* \in A^2$ such that $xx^* = e$.

We define the multilinear map $\mu_M : A^1 \times A^1 \times A^1 \to \mathbb{F}$ by $\mu_M(x, y, z) = (x \smile y \smile z)([M])$. Then, (2.1) for $i = j = 1$ gives us that $\mu_M$ is an alternating 3-form. In the above basis, if $x \smile y \smile z = \eta e$, then $\mu_M(x, y, z) = \eta$.

**Proposition 2.** *The 3-form $\mu_M$ and Poincaré duality determine the cup product $A^1 \times A^1 \to A^2$.*

*Proof.* Take $n = 3$ and $p = 2$ in Theorem 4. To each $\alpha, \beta \in A^1$ corresponds an element of $\mathrm{Hom}_{\mathbb{F}}(A^1, \mathbb{F})$ defined by sending $x \mapsto \mu(\alpha, \beta, x)$. Thus $\alpha \smile \beta \in A^2$ is $\varphi^{-1}(\mu(\alpha, \beta, \cdot))$. ■

In the basis for $A^1$ of Proposition 1 (for $n = 3$ and $p = 1$), an arbitrary 3-form can be written as, for scalars $a_{ijk} \in \mathbb{F}$,

$$\mu = \sum_{i<j<k} a_{ijk}dx^i \wedge dx^j \wedge dx^k. \tag{2.2}$$

In the above basis, we have the formula $x_i \smile x_j = \sum_k a_{ijk}\overline{x_k} \in A^2$, justified by Proposition 2.2.

Define the evaluation map $\iota_\mu : A^1 \to \Lambda^2(A^1)$ by $\iota_\mu(x) = \mu(x, \cdot, \cdot)$.

**Example 1.** The 3-torus $T = S^1 \times S^1 \times S^1$ has first Betti number 3 and three-form $\mu_T = dx^1 \wedge dx^2 \wedge dx^3$. Its cohomology algebra is then the exterior algebra $\Lambda^3(\mathbb{F})$, and as a result $\ker(\iota_\mu) = \{0\}$.

**Example 2.** Let $M = \#^\beta(S^1 \times S^2)$, the connected sum of $\beta$ copies of $S^1 \times S^2$. The Künneth formula, which can be found in [5, Section 3.2] for example, gives us an isomorphism $H^*(S^1 \times S^2;\mathbb{F}) \cong H^*(S^1;\mathbb{F}) \otimes H^*(S^2;\mathbb{F})$. This gives us $H^1(S^1 \times S^2;\mathbb{F}) \cong H^2(S^1 \times S^2;\mathbb{F}) \cong \mathbb{F}$. Suppose $a$ generates $H^1$ and $b$ generates $H^2$. Then $a \smile b$ generates $H^3$. It is standard to show that taking the connected sum preserves the cup product structure on each copy of $S^1 \times S^2$ and sets cup products of cohomology classes from different copies to 0; see for example [4, Chapter VI, Section 9]. This results in the cup product on $H^1(M;\mathbb{F}) \cong \mathbb{F}^\beta$ being trivial, giving $\mu_M = 0$ and thus $\iota_\mu = 0$ and $\ker(\iota_\mu) = A^1$.

### 2.3 Hochschild cohomology

For a field $\mathbb{F}$, Hochschild cohomology associates a sequence of $\mathbb{F}$-vector spaces $HH^i(A)$ to an $\mathbb{F}$-algebra $A$. In Hochschild's original paper [6], the Hochschild chain complex of $A$ with coefficients in $A$ are defined as

$$CC^k(A) = \mathrm{Hom}_{\mathbb{F}}(A^{\otimes k}, A)$$

where $A^{\otimes k}$ is the tensor product of $A$ with itself $k$ times and $A^{\otimes 0} = \mathbb{F}$. They are equipped with the differential $d : CC^k(A) \to CC^{k+1}(A)$ defined by the following formula, for $f \in CC^k(A)$:

$$\begin{aligned} df(a_1 \otimes \cdots \otimes a_{k+1}) &= a_1 f(a_2 \otimes \cdots \otimes a_{k+1}) \\ &+ \sum_{i=1}^{k}(-1)^i f(a_1 \otimes \cdots \otimes a_ia_{i+1} \otimes \cdots \otimes a_{k+1}) \\ &+ (-1)^{k+1}f(a_1 \otimes \cdots \otimes a_k)a_{k+1}. \end{aligned} \tag{2.3}$$

For the $k = 0$ case, we have $df(a_1) = a_1f(1) - f(1)a_1$.

The proof of the following proposition is a tedious calculation that will be omitted. It can be found in [6].

**Proposition 3.** $d^2 = 0$.

Thus we can define the *$n$-th Hochschild cohomology of $A$* (with coefficients in $A$) as

$$HH^n(A) = \frac{\ker(d : CC^n(A) \to CC^{n+1}(A))}{\mathrm{im}(d : CC^{n-1}(A) \to CC^n(A))}.$$

Note that for $n \leq -1$, $HH^n(A) = 0$.

### 2.4 Bigraded Hochschild cohomology

Let $A = \oplus_i A^i$ be a graded algebra. A standard procedure, described in for example [7, Section 5.4], is incorporating the grading of $A$ into its Hochschild cohomology by defining the bigraded Hochschild complex $CC^{n,r}(A) = \mathrm{Hom}^r_{\mathbb{F}}(A^{\otimes n}, A) \subset CC^n(A)$. Here $\mathrm{Hom}^r_{\mathbb{F}}(A^{\otimes n}, A)$ is the set of all maps $f \in \mathrm{Hom}_{\mathbb{F}}(A^{\otimes n}, A)$ such that

$$|f(a_1 \otimes \cdots \otimes a_n)| = \sum_{i=1}^{n} |a_i| + r.$$

We can verify that $d(CC^{n,r}(A)) \subset CC^{n+1,r}$, that is, the differential $d$ preserves the grading. Thus we can define $HH^{*,r}(A)$, the bigraded Hochschild cohomology of degree $r$, by

$$HH^{n,r}(A) = \frac{\ker(d : CC^{n,r} \to CC^{n+1,r})}{\mathrm{im}(d : CC^{n-1,r} \to CC^{n,r})}. \tag{2.4}$$

### 2.5 Quadratic forms

Let $A = H^*(L; \mathbb{C})$ be the cohomology algebra of a closed orientable 3-manifold $L$. Biran and Cornea [8, Section 5.3] define a map $\Theta : HH^{2,-2}(A) \to Q^2(A^1; \mathbb{C})$ to the space of quadratic forms on $A^1$ as follows. Consider an element $f \in CC^{2,-2}(A)$, and restrict it to a map $f : A^1 \otimes A^1 \to A^{1+1-2} \cong \mathbb{F}$. Define $\Theta(f) \in Q^2(A^1; \mathbb{C})$ to be the quadratic form $\Theta(f)(x) = f(x \otimes x)$. The proof of the following can be found in [8, Section 5.3.1].

**Proposition 4.** *The map $\Theta$ is well defined on cohomology classes in $HH^{2,-2}(A)$.*

*Proof.* It is sufficient to show that $\Theta = 0$ on coboundaries. Let $f \in CC^{2,-2}(A)$ be a coboundary $f = dg$. Let $x \in A^1$. Then $\Theta(f)(x) = f(x \otimes x) = dg(x \otimes x) = xg(x) - g(x \cdot x) + g(x)x$. But $x \cdot x = 0$ by (2.1) and $|g(x)| = |x| - 2 = -1$ since $g \in CC^{1,-2}(A)$. Therefore $\Theta(dg)(x) = 0$. ∎

The discriminant $\Delta$ that Charette considers in [1] is $\Delta(\psi)$ for a quadratic form $\psi \in \mathrm{im}\,\Theta$. Thus, if $HH^{2,-2}(A) = 0$, then $\mathrm{im}(\Theta) = 0$ and as a result $\Delta = 0$.

## 3 Zeroth Hochschild cohomology

The following is a standard result that can be found in [9, Section 9.1] for example.

**Proposition 5.** $HH^0(A) \cong Z(A)$*, the center of the algebra $A$.*

Let $A = \oplus_i A^i$ be a graded commutative Frobenius algebra of degree 3. For $a \in A^i$, we write its degree $|a| = i$.

By (2.1), we know that $A^0$ and $A^2$ are in $Z(A)$. In fact, $A^3 \subset Z(A)$ as well because the only nontrivial cup product with elements of $A^3$ is a commutative one with $A^0$. We have a lemma:

**Lemma 1.** *$x \in A^1$ is in $Z(A)$ if and only if $xyz = 0$ for all $y, z \in A^1$.*

*Proof.* Let $x \in A^1$ be in $Z(A)$. Then, for any $y \in A^1$, we have $xy = yx = -xy$ by graded commutativity. Then $2xy = 0$, which implies that $xy = 0$ since $\mathrm{char}(\mathbb{F}) \neq 2$. Therefore $xyz = 0$ for all $y, z \in A^1$.

Let $x \in A^1$ such that $xyz = 0$ for all $y, z$. Suppose that there exists $y$ such that $xy \neq 0$. Then, as previously mentioned, by Proposition 1, we can choose $z \in A^1$ dual to $xy$ in the sense that $xyz = e$. This contradicts the hypothesis that $xyz = 0$ for all $y, z$, so we must have $xy = 0$ for all $y \in A^1$. Therefore $xy = yx = 0$ for all $y \in A^1$ and $x \in Z(A)$. ∎

*Proof of Theorem 1.* Suppose that $x \in A^1$ is in $Z(A)$. Then, by Lemma 1, $xyz = 0$ for all $y, z \in A^1$. Then $\iota_\mu(x)(y, z) = \mu_A(x, y, z) = \sigma(x, yz) = \sigma(1, xyz) = \sigma(1, 0) = 0$ for all $y, z$ and thus $\iota_\mu(x) = 0$.

Conversely, suppose that $\iota_\mu(x) = 0$. Then $\mu_A(x, y, z) = \sigma(x, yz) = \sigma(1, xyz) = 0$ for all $y, z \in A^1$. By the nondegeneracy of $\sigma$ on $A^0 \times A^3$ and the fact that $A^3 \cong \mathbb{F}$, we must have $xyz = 0$ for all $y, z \in A^1$, so that $x \in Z(A)$ by Lemma 1.

Therefore, by Proposition 5, we have $HH^0(A) \cong A^0 \oplus \ker(\iota_\mu) \oplus A^2 \oplus A^3$. Note that $A^0 \cong A^3 \cong \mathbb{F}$, so that by counting dimensions, we get $HH^0(A) \cong \mathbb{F}^{2+\dim A^2} \oplus \ker(\iota_\mu)$. ∎

## 4 Bigraded Hochschild cohomology of an algebra with trivial 3-form

*Proof of Theorem 2.* We choose the same bases for the algebra $A$ as in Proposition 1 and (2.2). That is, we choose a basis $\{x_1, \cdots, x_\beta\}$ for $A^1$ and a basis $\{\overline{x_1}, \cdots, \overline{x_\beta}\}$ for $A^2$ such that $x_i\overline{x_j} = \delta_{ij}e$, where $e$ is a generator of $A^3$.

All products commute since the only noncommutative product in $A$ is $A^1 \times A^1 \to A^2$, which vanishes for $\mu = 0$. The product $A^0 \times A^i \to A^i$ is scalar multiplication, the product $A^1 \times A^2 \to A^3$ is, in the chosen basis, characterized by the relation $x_i\overline{x_j} = \delta_{ij}e$, and all other products $A^i \times A^j$ vanish.

We give a basis for $CC^{1,-2}(A) = \mathrm{Hom}^{-2}_{\mathbb{F}}(A, A)$. Define $f_p$ with $f_p(\overline{x_i}) = \delta_{ip}1 \in A^0$ and $f_p(e) = 0$, define $g_p$ with $g_p(\overline{x_i}) = 0$ and $g_p(e) = x_p$. We see that $\{f_1, \ldots, f_\beta, g_1, \ldots, g_\beta\}$ is a basis for $CC^{1,-2}(A)$.

We now describe the image of $d : CC^{1,-2} \to CC^{2,-2}$ (which is injective, giving $HH^{1,-2}(A) = 0$, but we don't need that fact). We have the differential $df(a_1 \otimes a_2) = a_1 f(a_2) - f(a_1a_2) + f(a_1)a_2$. By linearity it suffices to consider $a_i$ to be basis elements of $A$. Since $df(a_1 \otimes a_2) = df(a_2 \otimes a_1)$ by the fact that $A$ is commutative, it suffices to consider half the cases.

$df_p$ is nonzero only when either $a_1$ or $a_2$ is $\overline{x_p}$ and neither is 1. For suppose without loss of generality that $a_1 = 1$. Then $df(1 \otimes a_2) = f(a_2) - f(a_2) + f(1)a_2 = 0$. Then, suppose $a_1, a_2 \in A^2$. Then $df(a_1 \otimes a_2) = -f(a_1a_2) = 0$ by the fact that the product $A^1 \times A^1$ is trivial since $\mu = 0$. The only nonzero values $df_p$ can take in $A^0 \cup A^1$ are, up to multiplication by a scalar,

$$df_p(x_i \otimes \overline{x_p}) = x_i. \tag{4.1}$$

We move on to $dg_p(x_i \otimes \overline{x_j})$. The only nonzero values in $A^0 \cup A^1$, up to a scalar factor are

$$dg_p(x_i \otimes \overline{x_i}) = -x_p. \tag{4.2}$$

Equations (4.1) and (4.2) imply that for every $h = \sum_m(\alpha_m f_m + \gamma_m g_m) \in$

$CC^{1,-2}$, if $i,j,k$ are distinct, we have

$$
\begin{aligned}
dh(x_i \otimes \overline{x_j})\overline{x_k} &= \sum_m \alpha_m df_m(x_i \otimes \overline{x_j})\overline{x_k} + \gamma_m dg_m(x_i \otimes \overline{x_j})\overline{x_k} \\
&= 0.
\end{aligned} \tag{4.3}
$$

Assuming that $\dim(A^1) \geq 3$, we define $\varphi \in CC^{2,-2}(A)$ as follows:

$$\varphi(x_1 \otimes \overline{x_2}) = x_3, \varphi(x_1 \otimes \overline{x_3}) = x_2 \text{ and } \varphi(\overline{x_2} \otimes \overline{x_3}) = \overline{x_1}.$$

Set $\varphi$ to be symmetric, that is, such that $\varphi(a_1 \otimes a_2) = \varphi(a_2 \otimes a_1)$, and set $\varphi$ to zero on every other generator of $A \otimes A$. Clearly $\varphi \notin d(CC^{1,-2})$ by (4.3).

We show that $d\varphi = 0$ for all $a_1, a_2, a_3$ using the differential formula of (2.3):

$$
\begin{aligned}
d\varphi(a_1 \otimes a_2 \otimes a_3) &= a_1\varphi(a_2 \otimes a_3) - \varphi(a_1a_2 \otimes a_3) + \varphi(a_1 \otimes a_2a_3) \\
&\quad - \varphi(a_1 \otimes a_2)a_3.
\end{aligned} \tag{4.4}
$$

It is sufficient to only check for $a_i$ basis elements of $A$ by linearity. Furthermore, we only need to check one of $d\varphi(a_3 \otimes a_2 \otimes a_1) = 0$ and $d\varphi(a_1 \otimes a_2 \otimes a_3) = 0$ since $\varphi$ is symmetric.

It is clear that if any one of $a_1, a_2$ or $a_3$ is 1, then $d\varphi = 0$ because at least two terms of (4.4) cancel out and $\varphi(1 \otimes a_i) = 0$ by the definition of $\varphi$. It is also clear that $d\varphi(a_1 \otimes e \otimes a_3) = 0$ since we defined $\varphi$ such that $\varphi(a_i \otimes e) = 0$. We calculate

$$d\varphi(a_1 \otimes a_2 \otimes e) = -\varphi(a_1 \otimes a_2)e$$

which can only be nonzero when $\varphi(a_1 \otimes a_2) \neq 0$ in $A^0$, which never occurs since $\varphi$ was defined to be zero on all generators $a_1 \otimes a_2$ such that $|a_1| + |a_2| = 2$. Therefore, if any one of $a_1, a_2, a_3$ is $e$, $df = 0$. For this reason, from this point on we take $a_i \in A^1 \cup A^2$.

We have

$$d\varphi(a_1 \otimes x_j \otimes a_3) = a_1\varphi(x_j \otimes a_3) - \varphi(a_1 \otimes x_j)a_3.$$

If either of $a_1$ or $a_3$ is in $A^1$, this expression is 0 because $\varphi(x_i \otimes x_j) = 0$, $\varphi(1 \otimes x_i) = 0$, and $\mu_A = 0$. We compute

$$d\varphi(x_i \otimes \overline{x_j} \otimes x_k) = x_i\varphi(\overline{x_j} \otimes x_k) - \varphi(x_i \otimes \overline{x_j})x_k = 0$$

by the fact that the product $A^1 \times A^1 \to A^2$ is trivial since $\mu = 0$. We have $d\varphi(\overline{x_i} \otimes \overline{x_j} \otimes \overline{x_k}) \in A^4 = 0$. Therefore, we have two last cases to check:

$$d\varphi(\overline{x_i} \otimes x_j \otimes \overline{x_k}) = \overline{x_i}\varphi(x_j \otimes \overline{x_k}) - \varphi(\overline{x_i} \otimes x_j)\overline{x_k} = 0, \tag{4.5}$$

$$d\varphi(x_i \otimes \overline{x_j} \otimes \overline{x_k}) = x_i\varphi(\overline{x_j} \otimes \overline{x_k}) - \varphi(x_i \otimes \overline{x_j})\overline{x_k} = 0. \tag{4.6}$$

Both (4.5) and (4.6) are true if $i, j, k$ are $\geq 4$. Note that only one of the cases $(i,j,k)$ and $(k,j,i)$ needs to be checked. By (4.6) and by the way $\varphi$ was defined, it is sufficient to check the cases in which $i,j,k$ are distinct. Checking by hand over $(i,j,k) = \{(1,2,3),(2,1,3),(1,3,2)\}$ we see that both equations are always satisfied.

Thus, $\varphi \in \ker(d : CC^{2,-2} \to CC^{3,-2})$ but $\varphi \notin d(CC^{1,-2})$ and as a result $HH^{2,-2}(A) \neq 0$. ∎

## Acknowledgments

I would like to thank my mentor and supervisor Dr. François Charette of Marianopolis College for providing teaching and guidance for the completion of this project. The project was funded by Dr. Charette's grant #271306 from the *Fonds de recherche du Québec - Nature et technologies (FRQNT)*.

I also thank the anonymous referees who provided detailed comments and criticism, which helped improve the structure and content of this article.

## References

1. Charette F. On the cohomology ring of narrow Lagrangian 3-manifolds, quantum Reidemeister torsion, and the Laudau-Ginzburg superpotential. Arxiv. [Preprint] 2016. Available from https://arxiv.org/abs/1603.05586.
2. Sullivan D. On the intersection ring of compact three manifolds. Topology. 1975;14(3):275-277.
3. Hoffman K, Kunze R. Linear Algebra. Englewood Cliffs: Prentice-Hall; 1971. 407p.
4. Bredon G. Topology and Geometry. New York: Springer-Verlag; 1993. 557 p.
5. Hatcher A. Algebraic Topology. Cambridge: Cambridge University Press; 2001. 551 p.
6. Hochschild G. On the cohomology groups of an associative algebra. Annals of Mathematics. 1945;46(1):58-67.
7. Witherspoon, S. J. Hochschild Cohomology for Algebras. Providence: American Mathematical Society; 2019. 251p.
8. Biran P, Cornea O. Lagrangian topology and enumerative geometry. Geometry & Topology. 2012;16(2):963-1052.
9. Weibel C. A. An introduction to homological algebra. Cambridge: Cambridge University Press; 1994. 450 p.