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Research Article

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Improved Hardy-Sobolev Inequality under Moment Constraints

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Abstract

This paper is inspired by Aubin's 1979 result, which established that the best constant in the Sobolev inequality on the n-sphere, $\mathbb{S}^n$, can be improved under the condition of vanishing first-order moments. Recent advancements by Hang and Wang (2021) showed that Aubin's improvement can be generalized to arbitrary higher-order moments. We further extend Hang and Wang's results to the Hardy-Sobolev inequality on $\mathbb{S}^n$ by deriving an associated concentration-compactness principle and imposing similar moment constraints. Finally, we briefly outline a framework for extending these results to higher-order Sobolev spaces.

Introduction & Context

Sobolev Inequalities

Sobolev inequalities are fundamental tools in the study of mathematical analysis, geometry, and partial differential equations. They play a crucial role in embedding theorems, isoperimetric inequalities, and in ensuring the existence, uniqueness, and regularity of solutions to partial differential equations. Essentially, they establish a relationship between the L^p norms of functions and their derivatives, allowing one to trade regularity for integrability: a function that is sufficiently smooth (regularity) in L^p is also guaranteed to belong to a higher L^q space (integrability), where $q > p$.

To precisely state the classical Sobolev inequality, we first introduce the necessary function spaces. Let $n \geq 2$ and let $1 \leq p < n$. Denote by $\mathcal{D}(\mathbb{R}^n) = C_c^\infty(\mathbb{R}^n)$ the space of smooth functions with compact support and let $\mathcal{D}^{1,p}(\mathbb{R}^n)$ be its completion under the norm

$$\|u\| = \left(\int_{\mathbb{R}^n} |\nabla u|^p\right)^{1/p}. \tag{1}$$

This paper will primarily focus on functions in the Sobolev space $W^{1,p}$, but we will state Euclidean theorems in the larger space $\mathcal{D}^{1,p}$ for greater generality. Denote by $p^* = np/(n-p)$ the critical Sobolev exponent. The classical Sobolev inequality on $\mathbb{R}^n$, as proven by Sobolev [1, 2], is as follows:

Theorem 1 (Euclidean Sobolev Inequality). *There exists a constant $C_{n,p} > 0$ such that, for any $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$,*

$$\left(\int_{\mathbb{R}^n} |u|^{p^*}\, dx\right)^{1/p^*} \leq C_{n,p} \left(\int_{\mathbb{R}^n} |\nabla u|^p\, dx\right)^{1/p}. \tag{2}$$

In the study of Sobolev inequalities, we are often interested in determining the smallest possible value of $C_{n,p}$ for which Theorem 1 remains valid. This minimal value for $C_{n,p}$ is often referred to as the best Sobolev constant. We will denote this best constant by $\mathbf{K}_{n,p}$. Rodemich [3], Aubin [4], and Talenti [5] proved that the best constant for $C_{n,p}$ exists and computed its value:

$$\mathbf{K}_{n,p} = \pi^{-1/2} n^{-1/p} \left(\frac{p-1}{n-p}\right)^{1-1/p} \left(\frac{\Gamma(n/2+1)\Gamma(n)}{\Gamma(n/p)\,\Gamma(n-n/p+1)}\right)^{1/n}.$$

Aubin [6] later extended the Euclidean Sobolev inequality (Theorem 1) to smooth, compact, Riemannian manifolds without boundary. The result is as follows:

Theorem 2 (Riemannian Sobolev Inequality). *Let (M, g) be a smooth, closed, Riemannian n-manifold. Let $1 \leq p < n$ and let $p^* = np/(n-p)$. Then, for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ that depends only on ε, M, and g such that, for any $u \in W^{1,p}(M)$,*

$$\left(\int_M |u|^{p^*}\, dv_g\right)^{p/p^*} \leq (\mathbf{K}_{n,p}^p + \varepsilon) \int_M |\nabla_g u|^p\, dv_g + C_\varepsilon \int_M |u|^p\, dv_g. \tag{3}$$

where ∇_g is the gradient with respect to the metric g and dv_g is the volume form on M.

Note that, for any Riemannian manifold, the Sobolev constant $\mathbf{K}_{n,p}^p + \varepsilon$ can be made arbitrarily close to $\mathbf{K}_{n,p}^p$ by choosing $\varepsilon > 0$ to be sufficiently small. However, unlike in the Euclidean case, the best Sobolev constant $\mathbf{K}_{n,p}^p$ cannot be achieved without the accompanying constant C_ε diverging.

Let $\mathbb{S}^n$ denote the n-sphere, the n-dimensional generalization of the 1-dimensional circle and the 2-dimensional sphere to any non-negative integer n. Aubin [7] showed that the Sobolev constant $\mathbf{K}_{n,p}^p + \varepsilon$ on $\mathbb{S}^n$ can be improved to $\mathbf{K}_{n,p}^p / 2^{p/n} + \varepsilon$ under the constraint that the first-order moments of $|u|^{p^*}$ vanish. In other words, if the function $|u|^{p^*}$ satisfies certain symmetry conditions, the value of the Sobolev constant $\mathbf{K}_{n,p}^p + \varepsilon$ can be lowered to $\mathbf{K}_{n,p}^p / 2^{p/n} + \varepsilon$. A precise statement of this result is as follows:

Theorem 3 (Aubin [7]). *Let $(\mathbb{S}^n, g_0)$ denote the n-sphere equipped with the round metric. Let $1 < p < n$ and let $p^* = np/(n-p)$. Then, for any*

$\varepsilon > 0$, *there exists a constant* $C_\varepsilon > 0$ *that depends only on* ε *such that*

$$\left(\int_{\mathbb{S}^n} |u|^{p^*}\, dv_{g_0}\right)^{p/p^*} \leq \left(\frac{\mathbf{K}_{n,p}^p}{2^{p/n}} + \varepsilon\right) \int_{\mathbb{S}^n} |\nabla u|^p\, dv_{g_0} + C_\varepsilon \int_{\mathbb{S}^n} |u|^p\, dv_{g_0}, \tag{4}$$

for any $u \in W^{1,p}(\mathbb{S}^n)$ *that satisfies*

$$\int_{\mathbb{S}^n} x_i |u|^{p^*}\, dv_{g_0} = 0 \tag{5}$$

for $i = 1, 2, ..., n+1$, *where* $(x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1}$.

Hang and Wang [8] further generalized Theorem 3 to higher-order moments. For consistency, we will follow their notation. In particular, denote by $\mathring{\mathcal{P}}_m$ the set of all polynomials $f : \mathbb{R}^{n+1} \to \mathbb{R}$ with degree at most m such that

$$\int_{\mathbb{S}^n} f\, dv_{g_0} = 0. \tag{6}$$

Additionally, for $0 < \theta < 1$ and $m \in \mathbb{N}$, define

$$\Theta(m, \theta, n) = \inf \left\{ \sum_i \nu_i^\theta \; : \; \nu \text{ probability measure supported on countably many points } \{\xi_i\} \subseteq \mathbb{S}^n \text{ such that } \int_{\mathbb{S}^n} f\, d\nu = 0 \text{ for all } f \in \mathring{\mathcal{P}}_m, \nu_i := \nu(\{\xi_i\}) \right\}. \tag{7}$$

Hang and Wang's [8] generalization of Aubin's improvement (Theorem 3) to higher-order moments is then as follows:

Theorem 4 (Hang and Wang [8])**.** *Let* $(\mathbb{S}^n, g_0)$ *denote the* n*-sphere equipped with the round metric. Let* $1 < p < n$, *let* $m \in \mathbb{N}$, *and let* $p^* = np/(n-p)$. *Then, for any* $\varepsilon > 0$, *there exists a constant* $C_\varepsilon > 0$ *that depends only on* ε *such that*

$$\left(\int_{\mathbb{S}^n} |u|^{p^*}\, dv_{g_0}\right)^{p/p^*} \leq \left(\frac{\mathbf{K}_{n,p}^p}{\Theta(m, p/p^*, n)} + \varepsilon\right) \int_{\mathbb{S}^n} |\nabla u|^p\, dv_{g_0} + C_\varepsilon \int_{\mathbb{S}^n} |u|^p\, dv_{g_0}, \tag{8}$$

for any $u \in W^{1,p}(\mathbb{S}^n)$ *that satisfies*

$$\int_{\mathbb{S}^n} f |u|^{p^*}\, dv_{g_0} = 0 \tag{9}$$

for all $f \in \mathring{\mathcal{P}}_m$.

Hang and Wang [8] also showed that $\Theta(1, p/p^*, n) = 2^{p/n}$ when $m = 1$, recovering Aubin's [7] original result (Theorem 3).

Hardy-Sobolev Inequalities

The Hardy-Sobolev inequality extends the Sobolev inequality to cases involving weighted integrals, where the integrand is multiplied by a weight function. In this paper, we will focus on weight functions of the form $d_g(x, x_0)^\alpha$, where $d_g(\cdot, x_0)$ denotes the Riemannian distance from a fixed point $x_0 \in M$. For the Euclidean case, this simplifies to $d_g(x, x_0)^\alpha = |x - x_0|^\alpha$, and we may set $x_0 = 0$ without loss of generality. Let $1 < p < n$, let $0 < \alpha < p$, and denote the critical Hardy-Sobolev exponent by $p_*(\alpha) = (n-\alpha)p/(n-p)$. Under these conditions, the Hardy-Sobolev inequality on $\mathbb{R}^n$ is as follows [9, 10]:

Theorem 5 (Euclidean Hardy-Sobolev Inequality)**.** *There exists a constant* $C_{n,p,\alpha} > 0$ *such that, for any* $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$,

$$\left(\int_{\mathbb{R}^n} \frac{|u|^{p^*(\alpha)}}{|x|^\alpha}\, dx\right)^{1/p^*(\alpha)} \leq C_{n,p,\alpha} \left(\int_{\mathbb{R}^n} |\nabla u|^p\, dx\right)^{1/p}. \tag{10}$$

Ghoussoub and Yuan [11] showed that the best constant for $C_{n,p,\alpha}$ can be attained. We denote this best constant by $\mathbf{K}_{n,p,\alpha}$. Egnell [12] computed its value, which is given by [13]

$$\mathbf{K}_{n,p,\alpha} = (n-\alpha)^{(n-p)/(n-\alpha)} \left(\frac{n-p}{p-1}\right)^{p-1} \times \left(\frac{\Gamma((p(n-\alpha)+p-n)/(p-\alpha))}{\Gamma(p(n-\alpha)/(p-\alpha))}\right)^{(p-\alpha)/(n-\alpha)} \times \left(\frac{n\pi^{n/2}\Gamma((n-\alpha)/(p-\alpha))}{\Gamma(n/2+1)}\right)^{(p-\alpha)/(n-\alpha)}. \tag{11}$$

Jaber [14], and Chen and Liu [15] later extended the Euclidean Hardy-Sobolev inequality (Theorem 5) to smooth, compact, Riemannian manifolds without boundary. We state their result:

Theorem 6 (Riemannian Hardy-Sobolev Inequality)**.** *Let* (M, g) *be a smooth, closed, Riemannian* n*-manifold and fix some* $x_0 \in M$. *Let* $1 < p < n$, *let* $0 < \alpha < p$, *and let* $p^*(\alpha) = (n-\alpha)p/(n-p)$. *Then, for any* $\varepsilon > 0$, *there exists a constant* $C_\varepsilon > 0$ *that depends only on* ε, M, *and* g *such that, for any* $u \in W^{1,p}(M)$,

$$\left(\int_M \frac{|u|^{p^*(\alpha)}}{d_g(x, x_0)^\alpha}\, dv_g\right)^{p/p^*(\alpha)} \leq (\mathbf{K}_{n,p,\alpha}^p + \varepsilon) \int_M |\nabla_g u|^p\, dv_g + C_\varepsilon \int_M |u|^p\, dv_g, \tag{12}$$

where d_g *is the Riemannian distance on* (M, g).

The goal of this paper is to improve the Hardy-Sobolev inequality (Theorem 6) on $\mathbb{S}^n$ by imposing moment constraints analogous to those used by Hang and Wang [8] to improve the standard Sobolev inequality.

Main Result

The following result presents an extension of Theorem 4 to the Hardy-Sobolev inequality:

Theorem 7. *Let* $(\mathbb{S}^n, g_0)$ *denote the* n*-sphere equipped with the round metric and fix some* $x_0 \in \mathbb{S}^n$. *Let* $1 < p < n$, *let* $0 < \alpha < p$, *let* $m \in \mathbb{N}$, *and let* $p^*(\alpha) = (n-\alpha)p/(n-p)$. *Then, for any* $\varepsilon > 0$, *there exists a constant* $C_\varepsilon > 0$ *that depends only on* ε *such that*

$$\left(\int_{\mathbb{S}^n} \frac{|u|^{p^*(\alpha)}}{d(x, x_0)^\alpha}\, dv_{g_0}\right)^{p/p^*(\alpha)} \leq \left(\frac{\mathbf{K}_{n,p,\alpha}^p}{\Theta(m, p/p^*(\alpha), n)} + \varepsilon\right) \times \int_{\mathbb{S}^n} |\nabla u|^p\, dv_{g_0} + C_\varepsilon \int_{\mathbb{S}^n} |u|^p\, dv_{g_0}, \tag{13}$$

for any $u \in W^{1,p}(\mathbb{S}^n)$ *that satisfies*

$$\int_{\mathbb{S}^n} f \frac{|u|^{p^*(\alpha)}}{d(x, x_0)^\alpha}\, dv_{g_0} = 0 \tag{14}$$

for all $f \in \mathring{\mathcal{P}}_m$.

To prove this theorem, we take inspiration from Hang and Wang [8] and instead prove the following, more general statement:

Theorem 8. *Let $(\mathbb{S}^n, g_0)$ denote the n-sphere equipped with the round metric and fix some $x_0 \in \mathbb{S}^n$. Let $1 < p < n$, let $0 < \alpha < p$, let $m \in \mathbb{N}$, let $p^*(\alpha) = (n-\alpha)p/(n-p)$, and let $T : \mathring{\mathcal{P}}_m \to \mathbb{R}_{>0}$ be some map. Then, for any $\varepsilon > 0$, there exists a constant $C_{\varepsilon,T} > 0$ that depends only on ε and T such that*

$$\left(\int_{\mathbb{S}^n} \frac{|u|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha}\, dv_{g_0}\right)^{p/p^*(\alpha)} \leq \left(\frac{\mathbf{K}^p_{n,p,\alpha}}{\Theta(m, p/p^*(\alpha), n)} + \varepsilon\right) \times \int_{\mathbb{S}^n} |\nabla u|^p\, dv_{g_0} + C_{\varepsilon,T} \int_{\mathbb{S}^n} |u|^p\, dv_{g_0}, \quad (15)$$

for any $u \in W^{1,p}(\mathbb{S}^n)$ that satisfies

$$\int_{\mathbb{S}^n} f \frac{|u|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha}\, dv_{g_0} \leq T(f) \left(\int_{\mathbb{S}^n} |u|^p\, dv_{g_0}\right)^{p^*(\alpha)/p} \quad (16)$$

for all $f \in \mathring{\mathcal{P}}_m$.

Since any $u \in W^{1,p}(\mathbb{S}^n)$ satisfying Equation 14 automatically satisfies Equation 16, Theorem 7 is recovered by taking the limit as $T(f) \to 0$.

A Concentration-Compactness Principle

The proof of Theorem 8 requires a concentration-compactness principle, similar to those introduced by Lions [10, 16]. The concentration-compactness principle is a key tool in the calculus of variations designed to address the lack of compactness in infinite-dimensional function spaces such as Sobolev spaces. Unlike in finite-dimensional spaces, where every bounded sequence has a convergent subsequence, sequences in infinite-dimensional spaces may fail to converge due to their mass either concentrating at specific points or escaping to infinity. The concentration-compactness principle provides a precise description of this phenomenon: any loss of compactness is restricted to a discrete, at most countable set of points. This weaker notion of compactness is often sufficient to analyze sequences that would otherwise be too irregular to handle. The classical concentration-compactness principle on $\mathbb{R}^n$ is given by Lemma I.1 in Lions [16]. A variant of this principle specifically associated to the Euclidean Hardy-Sobolev inequality is given by the following result:

Theorem 9 (Lemma 2.4 in Lions [10])**.** *Let $1 < p < n$ and let (u_k) be a bounded sequence in $\mathcal{D}^{1,p}(\mathbb{R}^n)$. Suppose that $u_k \to u$ pointwise almost everywhere, the sequence of measures $(|\nabla u_k|^p\, dx)$ is tight, and*

$$|\nabla u_k|^p\, dx \rightharpoonup \mu, \quad (17)$$

$$\frac{|u_k|^{p^*(\alpha)}}{|x|^\alpha}\, dx \rightharpoonup \nu, \quad (18)$$

where μ and ν are some measures on $\mathbb{R}^n$. Then there exists a non-negative real number $\nu_0 \in \mathbb{R}$ such that

$$\nu = \frac{|u|^{p^*(\alpha)}}{|x|^\alpha}\, dx + \nu_0 \delta_0, \quad (19)$$

$$\mu \geq |\nabla u|^p\, dx + \mathbf{K}^{-p}_{n,p,\alpha} \nu_0^{p/p^*(\alpha)} \delta_0. \quad (20)$$

We require a generalization of Theorem 9 to smooth, closed, Riemannian manifolds. The result is as follows:

Theorem 10. *Let (M, g) be a smooth, closed, Riemannian n-manifold and fix some $x_0 \in M$. Let $1 < p < n$ and let (u_k) be a bounded sequence in $W^{1,p}(M)$. Suppose that $u_k \to u$ pointwise almost everywhere and that*

$$|\nabla_g u_k|^p\, dv_g \rightharpoonup \mu, \quad (21)$$

$$\frac{|u_k|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha}\, dv_g \rightharpoonup \nu, \quad (22)$$

where μ and ν are some measures on M. Then there exists a non-negative real number $\nu_0 \in \mathbb{R}$ such that

$$\nu = \frac{|u|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha}\, dv_g + \nu_0 \delta_{x_0}, \quad (23)$$

$$\mu \geq |\nabla_g u|^p\, dv_g + \mathbf{K}^{-p}_{n,p,\alpha} \nu_0^{p/p^*(\alpha)} \delta_{x_0}. \quad (24)$$

Before we prove this concentration-compactness principle, we first state two lemmas. The first is a useful inequality, and the second is related to concentration-compactness on manifolds.

Lemma 1. *Let $x, y \in \mathbb{R}$, and let $a \geq 1$. Then,*

$$||x|^a - |y|^a| \leq a(|x|^{a-1} + |y|^{a-1})|x - y|. \quad (25)$$

Proof. The proof is trivial if either $x = 0$ or $y = 0$. Suppose then that $x \neq 0$ and $y \neq 0$. Consider the function $f(t) = t^a$. By the mean value theorem, there exists a $z \in \mathbb{R}$ between $|x|$ and $|y|$ such that

$$\begin{aligned} ||x|^a - |y|^a| &= |az^{a-1}(|x| - |y|)| \\ &= az^{a-1}||x| - |y||. \end{aligned}$$

Since the map $t \mapsto t^{a-1}$ is increasing for $a \geq 1$ and z lies between $|x|$ and $|y|$, it follows that

$$\begin{aligned} ||x|^a - |y|^a| &\leq a \max\{|x|^{a-1}, |y|^{a-1}\}||x| - |y|| \\ &\leq a(|x|^{a-1} + |y|^{a-1})||x| - |y||. \end{aligned}$$

By the reverse triangle inequality,

$$||x|^a - |y|^a| \leq a(|x|^{a-1} + |y|^{a-1})|x - y|.$$

∎

Lemma 2 (Lions [16])**.** *Let (M, g) be a smooth, closed, Riemannian n-manifold. Let μ and ν be two bounded, non-negative measures on M and let $1 \leq p < q \leq \infty$. Suppose that there exists a constant $C_0 \geq 0$ such that, for any $\varphi \in C_c^\infty(M)$, the measures satisfy*

$$\left(\int_M |\varphi|^q\, d\nu\right)^{1/q} \leq C_0 \left(\int_M |\varphi|^p\, d\mu\right)^{1/p}. \quad (26)$$

Then there exists an at-most countable set $\mathcal{I}$, a set of distinct points $\{x_i\}_{i\in\mathcal{I}}$ in M, and positive numbers $\{\nu_i\}_{i\in\mathcal{I}}$ such that

$$\nu = \sum_{i\in\mathcal{I}} \nu_i \delta_{x_i}, \quad (27)$$

$$\mu \geq C_0^{-p} \sum_{i\in\mathcal{I}} \nu_i^{p/q} \delta_{x_i}. \quad (28)$$

In particular,

$$\sum_{i\in\mathcal{I}} \nu_i^{p/q} < \infty. \quad (29)$$

Although Lions [16] proved the above lemma in $\mathbb{R}^n$, it can be easily adapted to manifolds since the proof in $\mathbb{R}^n$ does not make use of any properties unique to Euclidean space. We now prove Theorem 10:

Proof of Theorem 10. Let $v_k := u_k - u$. Then $v_k \to 0$ pointwise almost everywhere. Suppose also that

$$|\nabla_g v_k|^p \, dv_g \rightharpoonup \widetilde{\mu}, \tag{30}$$

$$\frac{|v_k|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha} \, dv_g \rightharpoonup \widetilde{\nu}, \tag{31}$$

where $\widetilde{\mu}$ and $\widetilde{\nu}$ are some bounded, non-negative measures on M.

We first prove the condition on ν. Let $\varphi \in C_c^\infty(M)$. Then $\varphi v_k \in W^{1,p}(M)$. By the Riemannian Hardy-Sobolev inequality (Theorem 6),

$$\left(\int_M \frac{|\varphi v_k|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha} \, dv_g\right)^{p/p^*(\alpha)} \le (\mathbf{K}^p_{n,p,\alpha} + \varepsilon) \int_M |\nabla_g(\varphi v_k)|^p \, dv_g + C_\varepsilon \int_M |\varphi v_k|^p \, dv_g. \tag{32}$$

In the limit as $k \to \infty$, the left-hand side of Equation 32 goes to

$$\lim_{k\to\infty} \left(\int_M \frac{|\varphi v_k|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha} \, dv_g\right)^{p/p^*(\alpha)} = \left(\int_M |\varphi|^{p^*(\alpha)} \, d\widetilde{\nu}\right)^{p/p^*(\alpha)}.$$

We estimate the first term on the right-hand side of Equation 32 using Minkowski's inequality:

$$\begin{aligned}\left(\int_M |\nabla_g(\varphi v_k)|^p \, dv_g\right)^{1/p} &\le \left(\int_M (|\nabla_g\varphi||v_k| + |\varphi||\nabla_g v_k|)^p \, dv_g\right)^{1/p} \\ &\le \left(\int_M |\nabla_g \varphi|^p |v_k|^p \, dv_g\right)^{1/p} \\ &\quad + \left(\int_M |\varphi|^p |\nabla_g v_k|^p \, dv_g\right)^{1/p}.\end{aligned}$$

Then,

$$\begin{aligned}&\left|\left(\int_M |\nabla_g(\varphi v_k)|^p \, dv_g\right)^{1/p} - \left(\int_M |\varphi|^p|\nabla_g v_k|^p \, dv_g\right)^{1/p}\right| \\ &\quad\le \left(\int_M |\nabla_g\varphi||v_k|^p \, dv_g\right)^{1/p}.\end{aligned}$$

Since φ is smooth and has compact support, its gradient is bounded on M by some constant $C > 0$,

$$\begin{aligned}&\left|\left(\int_M |\nabla_g(\varphi v_k)|^p \, dv_g\right)^{1/p} - \left(\int_M |\varphi|^p|\nabla_g v_k|^p \, dv_g\right)^{1/p}\right| \\ &\quad\le C\left(\int_M |v_k|^p \, dv_g\right)^{1/p}.\end{aligned}$$

Since (v_k) is a bounded sequence in $W^{1,p}(M)$ and $v_k \to 0$ pointwise almost everywhere, then $v_k \rightharpoonup 0$ in $W^{1,p}(M)$. Since $1 < p < p^*$, it follows by the Rellich-Kondrachov theorem [17, 18] that $v_k \to 0$ in $L^p(M)$. Then, in the limit as $k \to \infty$, we have

$$\lim_{k\to\infty}\left|\left(\int_M |\nabla_g(\varphi v_k)|^p \, dv_g\right)^{1/p} - \left(\int_M |\varphi|^p|\nabla_g v_k|^p \, dv_g\right)^{1/p}\right| = 0.$$

Hence, the first term on the right-hand side of Equation 32 goes to

$$\begin{aligned}\lim_{k\to\infty}\int_M |\nabla_g(\varphi v_k)|^p \, dv_g &= \lim_{k\to\infty}\int_M |\varphi|^p|\nabla_g v_k|^p \, dv_g \\ &= \int_M |\varphi|^p \, d\widetilde{\mu}.\end{aligned}$$

Similarly, the second term on the right-hand side of Equation 32 goes to zero. Combining these results, we obtain

$$\left(\int_M |\varphi|^{p^*(\alpha)} \, d\widetilde{\nu}\right)^{1/p^*(\alpha)} \le (\mathbf{K}^p_{n,p,\alpha} + \varepsilon)^{1/p} \left(\int_M |\varphi|^p \, d\widetilde{\mu}\right)^{1/p}.$$

Since this is true for all $\varepsilon > 0$,

$$\left(\int_M |\varphi|^{p^*(\alpha)} \, d\widetilde{\nu}\right)^{1/p^*(\alpha)} \le \mathbf{K}_{n,p,\alpha} \left(\int_M |\varphi|^p \, d\widetilde{\mu}\right)^{1/p}. \tag{33}$$

It follows immediately from Lemma 2 that there exists an at-most countable set $\mathcal{I}$, a set of distinct points $\{x_i\}_{i\in\mathcal{I}}$ in M, and positive numbers $\{\widetilde{\nu}_i\}_{i\in\mathcal{I}}$ such that

$$\widetilde{\nu} = \sum_{i\in\mathcal{I}} \widetilde{\nu}_i \delta_{x_i}, \tag{34}$$

$$\widetilde{\mu} \ge \mathbf{K}^{-p}_{n,p,\alpha} \sum_{i\in\mathcal{I}} \widetilde{\nu}_i^{p/p^*(\alpha)} \delta_{x_i}. \tag{35}$$

And, in particular,

$$\sum_{i\in\mathcal{I}} \widetilde{\nu}_i^{p/p^*(\alpha)} < \infty. \tag{36}$$

Furthermore, let $\varphi \in C_c^\infty(M)$. By the triangle inequality,

$$\begin{aligned}&\left|\int_M \varphi\frac{|v_k|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha} \, dv_g - \int_M \varphi\frac{|u_k|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha} \, dv_g + \int_M \varphi\frac{|u|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha} \, dv_g\right| \\ &\quad\le \int_M |\varphi| \left|\frac{|v_k|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha} - \frac{|u_k|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha} + \frac{|u|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha}\right| \, dv_g.\end{aligned}$$

Since φ is smooth and has compact support, there exists a constant $C > 0$ such that

$$\begin{aligned}&\left|\int_M \varphi\frac{|v_k|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha} \, dv_g - \int_M \varphi\frac{|u_k|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha} \, dv_g + \int_M \varphi\frac{|u|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha} \, dv_g\right| \\ &\quad\le C\int_M \left|\frac{|v_k|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha} - \frac{|u_k|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha} + \frac{|u|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha}\right| \, dv_g.\end{aligned}$$

Consider the sequence (f_k) given by $f_k = |u_k|/d_g(x,x_0)^{\alpha/p^*(\alpha)}$. For simplicity, we also write $f = |u|/d_g(x,x_0)^{\alpha/p^*(\alpha)}$. Then,

$$\begin{aligned}&\left|\int_M \varphi\frac{|v_k|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha} \, dv_g - \int_M \varphi\frac{|u_k|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha} \, dv_g + \int_M \varphi\frac{|u|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha} \, dv_g\right| \\ &\quad\le C\int_M \left||f_k - f|^{p^*(\alpha)} - |f_k|^{p^*(\alpha)} + |f|^{p^*(\alpha)}\right| \, dv_g.\end{aligned}$$

Since $u_k \to u$ pointwise almost everywhere, it follows immediately that $f_k \to f$ pointwise almost everywhere. Observe also that, by the Riemannian Hardy-Sobolev inequality (Theorem 6),

$$\begin{aligned}\left(\int_M |f_k|^{p^*(\alpha)}\, dv_g\right)^{p/p^*(\alpha)} &= \left(\int_M \frac{|u_k|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha}\, dv_g\right)^{p/p^*(\alpha)} \\ &\le (\mathbf{K}^p_{n,p,\alpha}+\varepsilon)\int_M |\nabla_g u_k|^p\, dv_g \\ &\quad + C_\varepsilon \int_M |u_k|^p\, dv_g.\end{aligned}$$

Since (u_k) is a bounded sequence in $W^{1,p}(M)$, we find that (f_k) must also be a bounded sequence in $L^{p^*(\alpha)}$. Then, by the Brézis-Lieb lemma [19],

$$\lim_{k\to\infty}\int_M \left||f_k-f|^{p^*(\alpha)} - |f_k|^{p^*(\alpha)} + |f|^{p^*(\alpha)}\right|\, dv_g = 0.$$

Hence, in the limit as $k \to \infty$, we have

$$\begin{aligned}\lim_{k\to\infty}\Bigg|\int_M \varphi\frac{|v_k|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha}\, dv_g - \int_M \varphi\frac{|u_k|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha}\, dv_g \\ + \int_M \varphi\frac{|u|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha}\, dv_g\Bigg| = 0.\end{aligned}$$

Equivalently,

$$\widetilde{\nu} = \nu - \frac{|u|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha}\, dv_g.$$

Renaming each $\widetilde{\nu}_i$ to ν_i and rearranging the terms,

$$\nu = \frac{|u|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha}\, dv_g + \sum_{i\in\mathcal{I}} \nu_i\delta_{x_i}. \tag{37}$$

It remains to show that the set $\mathcal{I}$ is a singleton. Let $\varphi \in C_c^\infty(M)$ be such that $\text{supp}(\varphi) \subseteq M\backslash\{x_0\}$. By the triangle inequality,

$$\begin{aligned}&\left|\int_M \varphi\frac{|u_k|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha}\, dv_g - \int_M \varphi\frac{|u|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha}\, dv_g\right| \\ &\le \int_{\text{supp}(\varphi)} |\varphi| \left|\frac{|u_k|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha} - \frac{|u|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha}\right|\, dv_g.\end{aligned}$$

Since φ is smooth and has compact support, there exists a constant $C > 0$ such that

$$\begin{aligned}&\left|\int_M \varphi\frac{|u_k|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha}\, dv_g - \int_M \varphi\frac{|u|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha}\, dv_g\right| \\ &\le C\int_{\text{supp}(\varphi)} \left|\frac{|u_k|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha} - \frac{|u|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha}\right|\, dv_g.\end{aligned}$$

Since $\text{supp}(\varphi) \subseteq M\backslash\{x_0\}$ is compact and $d_g(x,x_0)$ is continuous and positive on $M\backslash\{x_0\}$, the function $1/d_g(x,x_0)^\alpha$ achieves a finite maximum on $\text{supp}(\varphi)$. We absorb this value into the constant C. Then,

$$\begin{aligned}&\left|\int_M \varphi\frac{|u_k|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha}\, dv_g - \int_M \varphi\frac{|u|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha}\, dv_g\right| \\ &\le C\int_{\text{supp}(\varphi)} \left||u_k|^{p^*(\alpha)} - |u|^{p^*(\alpha)}\right|\, dv_g.\end{aligned}$$

By applying Lemma 1 and absorbing a factor of $p^*(\alpha)$ into the constant C, we obtain

$$\begin{aligned}&\left|\int_M \varphi\frac{|u_k|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha}\, dv_g - \int_M \varphi\frac{|u|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha}\, dv_g\right| \\ &\le C\int_{\text{supp}(\varphi)} \left(|u_k|^{p^*(\alpha)-1} + |u|^{p^*(\alpha)-1}\right)|u_k-u|\, dv_g \\ &\le C\int_M |u_k|^{p^*(\alpha)-1}|u_k-u|\, dv_g \\ &\quad + C\int_M |u|^{p^*(\alpha)-1}|u_k-u|\, dv_g.\end{aligned}$$

By Hölder's inequality with Hölder conjugates $p^*(\alpha)/(p^*(\alpha)-1)$ and $p^*(\alpha)$, we obtain

$$\begin{aligned}&\left|\int_M \varphi\frac{|u_k|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha}\, dv_g - \int_M \varphi\frac{|u|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha}\, dv_g\right| \\ &\le C\left(\int_M |u_k|^{p^*(\alpha)}\, dv_g\right)^{(p^*(\alpha)-1)/p^*(\alpha)} \\ &\quad \times \left(\int_M |u_k-u|^{p^*(\alpha)}\, dv_g\right)^{1/p^*(\alpha)} \\ &\quad + C\left(\int_M |u|^{p^*(\alpha)}\, dv_g\right)^{(p^*(\alpha)-1)/p^*(\alpha)} \\ &\quad \times \left(\int_M |u_k-u|^{p^*(\alpha)}\, dv_g\right)^{1/p^*(\alpha)}.\end{aligned}$$

Since (u_k) is a bounded sequence in $W^{1,p}(M)$ and $1 < p^*(\alpha) < p^*$, it follows by the Rellich-Kondrachov theorem [17, 18], that (u_k) is also a bounded sequence in $L^{p^*(\alpha)}(M)$. We absorb this upper bound for $\|u_k\|_{p^*(\alpha)}^{p^*(\alpha)-1}$ into the constant C. Then,

$$\begin{aligned}&\left|\int_M \varphi\frac{|u_k|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha}\, dv_g - \int_M \varphi\frac{|u|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha}\, dv_g\right| \\ &\le C\left(\int_M |u_k-u|^{p^*(\alpha)}\, dv_g\right)^{1/p^*(\alpha)} \\ &\quad + C\left(\int_M |u|^{p^*(\alpha)}\, dv_g\right)^{(p^*(\alpha)-1)/p^*(\alpha)} \\ &\quad \times \left(\int_M |u_k-u|^{p^*(\alpha)}\, dv_g\right)^{1/p^*(\alpha)}.\end{aligned}$$

Since (u_k) is a bounded sequence in $L^{p^*(\alpha)}(M)$ and $u_k \to u$ pointwise almost everywhere, we obtain by Fatou's lemma,

$$\begin{aligned}\int_M |u|^{p^*(\alpha)}\, dv_g &\le \liminf_{k\to\infty}\int_M |u_k|^{p^*(\alpha)}\, dv_g \\ &\le C^{p^*(\alpha)}.\end{aligned}$$

Absorbing this upper bound for $\|u\|_{p^*(\alpha)}^{p^*(\alpha)-1}$ into the constant C,

$$\begin{aligned}&\left|\int_M \varphi\frac{|u_k|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha}\, dv_g - \int_M \varphi\frac{|u|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha}\, dv_g\right| \\ &\le C\left(\int_M |u_k-u|^{p^*(\alpha)}\, dv_g\right)^{1/p^*(\alpha)} \\ &\quad + C\left(\int_M |u_k-u|^{p^*(\alpha)}\, dv_g\right)^{1/p^*(\alpha)} \\ &\le C\left(\int_M |u_k-u|^{p^*(\alpha)}\, dv_g\right)^{1/p^*(\alpha)}.\end{aligned}$$

Since (u_k) is a bounded sequence in $W^{1,p}(M)$ and $u_k \to u$ pointwise almost everywhere, then $u_k \rightharpoonup u$ in $W^{1,p}(M)$. Since $1 < p^*(\alpha) < p^*$, it follows by the Rellich-Kondrachov theorem [17, 18] that $u_k \to u$ in $L^{p^*(\alpha)}(M)$. Then, in the limit as $k \to \infty$, we have

$$\begin{aligned}\lim_{k\to\infty} &\left| \int_M \varphi \frac{|u_k|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha}\, dv_g - \int_M \varphi \frac{|u|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha}\, dv_g \right| \\ &\le \lim_{k\to\infty} C \left(\int_M |u_k - u|^{p^*(\alpha)}\, dv_g \right)^{1/p^*(\alpha)} \\ &= 0.\end{aligned}$$

Since this holds for all $\varphi \in C_c^\infty(M)$ satisfying $\operatorname{supp}(\varphi) \subseteq M \backslash \{x_0\}$, it must be that $\nu_i = 0$ for all $i \in \mathcal{I}\backslash\{0\}$. Hence, we may write $\mathcal{I} = \{0\}$ and

$$\nu = \frac{|u|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha}\, dv_g + \nu_0 \delta_{x_0}. \tag{38}$$

We now prove the conditions on μ. First, let $\varphi \in C_c^\infty(M)$. By the triangle inequality,

$$\begin{aligned}&\left| \int_M \varphi |\nabla u_k|^p\, dv_g - \int_M \varphi |\nabla u|^p\, dv_g - \int_M \varphi |\nabla v_k|^p\, dv_g \right| \\ &\quad \le \int_M |\varphi|\, \big||\nabla u_k|^p - |\nabla u|^p - |\nabla v_k|^p\big|\, dv_g.\end{aligned}$$

Since φ is smooth and has compact support, there exists a constant $C > 0$ such that

$$\begin{aligned}&\left| \int_M \varphi |\nabla u_k|^p\, dv_g - \int_M \varphi |\nabla u|^p\, dv_g - \int_M \varphi |\nabla v_k|^p\, dv_g \right| \\ &\quad \le C \int_M \big||\nabla u_k|^p - |\nabla u|^p - |\nabla v_k|^p\big|\, dv_g \\ &\quad = C \int_M \big||\nabla u_k|^p - |\nabla u|^p - |\nabla u_k - \nabla u|^p\big|\, dv_g.\end{aligned}$$

Since (u_k) is a bounded sequence in $W^{1,p}(M)$, it follows that (∇u_k) is a bounded sequence in L^p. Then, by the Brézis-Lieb lemma [19],

$$\lim_{k\to\infty} \int_M \big||\nabla u_k|^p - |\nabla u|^p - |\nabla u_k - \nabla u|^p\big|\, dv_g = 0.$$

Hence, in the limit as $k \to \infty$, we have

$$\lim_{k\to\infty} \left| \int_M \varphi |\nabla u_k|^p\, dv_g - \int_M \varphi |\nabla u|^p\, dv_g - \int_M \varphi |\nabla v_k|^p\, dv_g \right| = 0.$$

Equivalently,

$$\begin{aligned}\mu &= |\nabla u|^p\, dv_g + \widetilde{\mu} \\ &\ge |\nabla u|^p\, dv_g + \mathbf{K}_{n,p,\alpha}^{-p} \sum_{i\in\mathcal{I}} \widetilde{\nu}_i^{p/p^*(\alpha)} \delta_{x_i} \\ &= |\nabla u|^p\, dv_g + \mathbf{K}_{n,p,\alpha}^{-p} \nu_0^{p/p^*(\alpha)} \delta_{x_0}.\end{aligned}$$

■

Proof of the Main Result

We now prove the main result, Theorem 8. For convenience, we restate it here:

Theorem. *Let $(\mathbb{S}^n, g_0)$ denote the n-sphere equipped with the round metric and fix some $x_0 \in \mathbb{S}^n$. Let $1 < p < n$, let $0 < \alpha < p$, let $m \in \mathbb{N}$, let $p^*(\alpha) = (n-\alpha)p/(n-p)$, and let $T : \mathring{\mathcal{P}}_m \to \mathbb{R}_{>0}$ be some map. Then, for any $\varepsilon > 0$, there exists a constant $C_{\varepsilon,T} > 0$ that depends only on ε and T such that*

$$\begin{aligned}\left(\int_{\mathbb{S}^n} \frac{|u|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha}\, dv_{g_0} \right)^{p/p^*(\alpha)} &\le \left(\frac{\mathbf{K}_{n,p,\alpha}^p}{\Theta(m, p/p^*(\alpha), n)} + \varepsilon \right) \\ &\quad \times \int_{\mathbb{S}^n} |\nabla u|^p\, dv_{g_0} \\ &\quad + C_{\varepsilon,T} \int_{\mathbb{S}^n} |u|^p\, dv_{g_0},\end{aligned} \tag{39}$$

for any $u \in W^{1,p}(\mathbb{S}^n)$ that satisfies

$$\int_{\mathbb{S}^n} f \frac{|u|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha}\, dv_{g_0} \le T(f) \left(\int_{\mathbb{S}^n} |u|^p\, dv_{g_0} \right)^{p^*(\alpha)/p} \tag{40}$$

for all $f \in \mathring{\mathcal{P}}_m$.

Proof of Theorem 8. For simplicity, define

$$\beta := \frac{\mathbf{K}_{n,p,\alpha}^p}{\Theta(m, p/p^*(\alpha), n)} + \varepsilon. \tag{41}$$

Suppose, towards a contradiction, that Equation 15 does not hold. Then, for any $k \in \mathbb{N}$, there exists a sequence $(u_k) \subseteq W^{1,p}(\mathbb{S}^n)$ satisfying

$$\int_{\mathbb{S}^n} f \frac{|u_k|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha}\, dv_{g_0} \le T(f) \left(\int_{\mathbb{S}^n} |u_k|^p\, dv_{g_0} \right)^{p^*(\alpha)/p} \tag{42}$$

such that

$$\begin{aligned}\left(\int_{\mathbb{S}^n} \frac{|u_k|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha}\, dv_{g_0} \right)^{p/p^*(\alpha)} &> \beta \int_{\mathbb{S}^n} |\nabla u_k|^p\, dv_{g_0} \\ &\quad + k \int_{\mathbb{S}^n} |u_k|^p\, dv_{g_0}.\end{aligned} \tag{43}$$

Since the left-hand side is finite by the Riemannian Hardy-Sobolev inequality, we may assume by rescaling that

$$\left(\int_{\mathbb{S}^n} \frac{|u_k|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha}\, dv_{g_0} \right)^{p/p^*(\alpha)} = 1. \tag{44}$$

It follows immediately that

$$\int_{\mathbb{S}^n} |\nabla u_k|^p\, dv_{g_0} \le \frac{1}{\beta}, \tag{45}$$

$$\int_{\mathbb{S}^n} |u_k|^p\, dv_{g_0} \le \frac{1}{k}. \tag{46}$$

Since (u_k) and (∇u_k) are both bounded in $L^p(\mathbb{S}^n)$, it follows that the sequence (u_k) is also bounded in $W^{1,p}(\mathbb{S}^n)$, a reflexive Banach space. Then, by the Banach-Alaoglu theorem, the sequence (u_k) has a weakly convergent subsequence in $W^{1,p}(\mathbb{S}^n)$, which we denote as (u_{k_j}). Suppose that $u_{k_j} \rightharpoonup u$ weakly for some $u \in W^{1,p}(\mathbb{S}^n)$. Then $u_{k_j} \rightharpoonup u$ weakly in $L^p(\mathbb{S}^n)$ and $\nabla u_{k_j} \rightharpoonup \nabla u$ weakly in $L^p(\mathbb{S}^n)$. However, we also have from

Equation 46 that $u_k \to 0$ in $L^p(\mathbb{S}^n)$, and hence $u_k \rightharpoonup 0$ weakly in $L^p(\mathbb{S}^n)$. By the uniqueness of the weak limit, it follows that $u \equiv 0$. Since it is clear that $u_{k_j} \to 0$ in $L^p(\mathbb{S}^n)$ also, by the Riesz-Fischer theorem, we may pass to a further subsequence, which we still denote as (u_{k_j}), that converges pointwise almost everywhere to zero. Now, consider the following sequences of measures:

$$(\mu_j) := \left(|\nabla u_{k_j}|^p \, dv_{g_0}\right), \tag{47}$$

$$(\nu_j) := \left(\frac{|u_{k_j}|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha} \, dv_{g_0}\right). \tag{48}$$

By Equations 44 and 45, these sequences satisfy $\sup_j \mu_j(K) < \infty$ and $\sup_j \nu_j(K) < \infty$ for each compact set $K \subseteq \mathbb{S}^n$. By the weak compactness for measures, there exist subsequences, which we still denote as (μ_j) and (ν_j), such that

$$\mu_j = |\nabla u_{k_j}|^p \, dv_{g_0} \rightharpoonup \mu, \tag{49}$$

$$\nu_j = \frac{|u_{k_j}|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha} \, dv_{g_0} \rightharpoonup \nu, \tag{50}$$

where μ and ν are some bounded, non-negative measures on $\mathbb{S}^n$. Hence, by the concentration-compactness principle (Theorem 10) there exists a non-negative real number $\nu_0 \in \mathbb{R}$ such that

$$\nu = \nu_0 \delta_{x_0}, \tag{51}$$

$$\mu \geq \mathbf{K}_{n,p,\alpha}^{-p} \nu_0^{p/p^*(\alpha)} \delta_{x_0}. \tag{52}$$

Furthermore,

$$\begin{aligned} \nu(\mathbb{S}^n) &= \int_{\mathbb{S}^n} d\nu \\ &= \lim_{k\to\infty} \int_{\mathbb{S}^n} \frac{|u_k|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha} \, dv_{g_0} \\ &= 1. \end{aligned}$$

And,

$$\begin{aligned} \mu(\mathbb{S}^n) &= \int_{\mathbb{S}^n} d\mu \\ &= \lim_{k\to\infty} \int_{\mathbb{S}^n} |\nabla u_k|^p \, dv_{g_0} \\ &\leq \frac{1}{\beta}. \end{aligned}$$

Then, for any $f \in \mathring{\mathcal{P}}_m$,

$$\begin{aligned} \left|\int_{\mathbb{S}^n} f \, d\nu\right| &= \left|\lim_{k\to\infty} \int_{\mathbb{S}^n} f \frac{|u_k|^{p^*(\alpha)}}{d_g(x,x_0)^\alpha} \, dv_{g_0}\right| \\ &\leq \lim_{k\to\infty} T(f) \left(\int_{\mathbb{S}^n} |u_k|^p \, dv_{g_0}\right)^{p^*(\alpha)/p} \\ &\leq \lim_{k\to\infty} T(f) \left(\frac{1}{k}\right)^{p^*(\alpha)/p} \\ &= 0. \end{aligned}$$

Therefore, ν satisfies the conditions presented in Equation 7. Then, by the definition of infimum,

$$\begin{aligned} \Theta(m, p/p^*(\alpha), n) &\leq \nu_0^{p/p^*(\alpha)} \\ &\leq \mathbf{K}_{n,p,\alpha}^p \mu(\mathbb{S}^n) \\ &\leq \frac{\mathbf{K}_{n,p,\alpha}^p}{\beta}. \end{aligned}$$

Hence,

$$\beta \leq \frac{\mathbf{K}_{n,p,\alpha}^p}{\Theta(m, p/p^*(\alpha), n)}. \tag{53}$$

This contradicts our definition of β as given in Equation 41. It must then be that Equation 15 holds. ∎

Conclusion

In this paper, we established an improved Hardy-Sobolev inequality on $\mathbb{S}^n$ under moment constraints, extending the work of Hang and Wang [8] from the standard Sobolev setting to the Hardy-Sobolev setting. To achieve this, we first derived a concentration-compactness principle adapted to the Hardy-Sobolev inequality on smooth, compact, Riemannian manifolds without boundary. Our main result demonstrates that imposing moment constraints on functions in $W^{1,p}(\mathbb{S}^n)$ leads to a tighter upper bound on the Hardy-Sobolev constant, similar to the improvements obtained by Aubin [7] and Hang and Wang [8] for the Sobolev case.

Following the approach of Hang and Wang [8] in Section 4 of their paper, a natural direction for future work is to extend the above results to higher-order Sobolev spaces $W^{k,p}(M)$. We briefly sketch the framework. Aubin [4] previously extended the Euclidean higher-order Sobolev inequality to smooth, compact, Riemannian manifolds without boundary. This suggests that a similar extension should hold for the higher-order Hardy-Sobolev inequality. Such an extension would allow us to generalize the concentration-compactness principle stated in Theorem 10 from $W^{1,p}(M)$ to $W^{k,p}(M)$. These results would then yield a higher-order version of Theorem 8, establishing a Hardy-Sobolev equivalent to Theorems 4.1 and 4.2 in Hang and Wang's paper [8].

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