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Research Article

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Description and Exploration of Mean-Gauss Surfaces

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Abstract

In this paper we explore solving the prescribed mean curvature equation for surfaces meeting a new relation given by $H_S = \lambda K_S$, where H_S and K_S are the mean and Gaussian curvatures, respectively. We prove several existence theorems for various families of surfaces and state a conjecture for surfaces of revolution. To conclude, we state a weak existence theorem, and a strong conjecture concerning possible solutions. The intention is that by using differential geometry tools which would have likely been seen at the undergraduate level, the paper and its results are more accessible. My hope is that these new theorems find applications in the classification of surfaces in the future, or at the very least serves as an interesting curiosity.

Introduction

We will be exploring the existence of surfaces for which a proportionality relationship between the mean and Gaussian curvatures exist. More specifically those for which the following holds:

$$H_S = \lambda K_S \tag{1}$$

where S is a smooth surface, H_S the mean curvature of S, K_S the Gaussian curvature, and λ a scalar. It would seem that solutions to this problem have not yet been researched, and as such, I will assign the name of Mean-Gauss surfaces to them.

To begin, we define the two curvatures and give explicit formulas for their computations. Then we proceed to show that there exists at least one Mean-Gauss surface (the sphere). Following which, we will work toward proving several results concerning the existence of other such surfaces, yielding several new theorems and conjectures.

The main body of exploration will be dealing with surfaces in $\mathbb{R}^3$ as these will allow for visualizations. Furthermore, since this problem does not seem to have been explored, beginning with the "simplest" case is likely a good place to start.

Throughout the paper, we assume only a surface level knowledge of differential geometry in the hopes that this work provides insights and inspiration to those not so far along their mathematical journey. Another thing to note is that this paper is an excerpt of a lager work that was submitted as my honours research project under the supervision of Dr. A. Stancu of Concordia University.

Mean-Gauss Surfaces

The prescribed mean curvature equation is given by:

$$div\left(\frac{\nabla u}{\sqrt{1+||\nabla u||^2}}\right) + f(u) = 0$$

where ∇ represents the gradient, and $u(x)$ is a function defining a surface. Furthermore, we have that:

$$div\left(\frac{\nabla u}{\sqrt{1+||\nabla u||^2}}\right)$$

is a scaled version of the mean curvature of an n dimensional manifold embedded in $\mathbb{R}^{n+1}$ defined as the graph $S = (x_1, x_2, ..., x_n, u(x_1, x_2, ..., x_n))$, hence its namesake. The proof of this fact is quite mechanical, but every part falls into place in a most satisfying manner; as such, I encourage the reader to attempt it. The inclusion of the divergence form here serves only to give an idea of what problem led me to investigating relationship (1), and will not be used past this section.

Possible solutions to the divergence equation is a topic with a large body of research behind it. Generally these take a differential equation approach, applying known techniques to find possible solutions, or prove the existence of solutions of specific forms.

For this paper, we will investigate whether there exist solutions when $f(u)$ is the Gaussian curvature of our surface or some scaled version of it. More specifically, we remove the requirement that our surface writes as a graph, and consider the more general relation $H_S = \lambda K_S$ instead.

Definition 1. *Define a Mean-Gauss surface to be a smooth surface S such that globally,*

$$H_S = \lambda K_S,$$

where K_S is the Gaussian curvature, H_S is the mean curvature, and $\lambda \in \mathbb{R}$. □

As for the question of why restrict λ to scalar values, we can consider the following:

Let S be a surface defined by $(x(u,v), y(u,v), z(u,v))$; then for $\lambda(u,v)$

we want

$$H_S = \lambda(u,v)K_S,$$

but if we allow λ to be a function, then we can simply choose

$$\lambda = \frac{H_S}{K_S}$$

which is a solution for all surfaces having $K_S \neq 0$ for any u, v in its domain. As such, this question does not require much investigating.

Even if $K_S = 0$ at some set of points $\{p_i\} \in S$, the relation still holds everywhere else on the surface. Furthermore, we have local satisfaction everywhere but on neighborhoods of the surface which are developable.

That being said, it would be interesting to ask for $\lambda(u,v)$ satisfying (1) and such that

$$T(\lambda(u,v)) \neq \frac{H_S}{K_S}$$

for any elementary or "simple" transformation T. This is a question not covered here, but which could prove to be an interesting avenue of future research.

Defining the Curvatures

We begin by giving a relation between the Weingarten map and the fundamental forms, from which we will define the curvatures to be used in the following sections. Going forward, we are dealing with surfaces embedded in $\mathbb{R}^3$ unless stated otherwise.

Weingarten Map

For a surface S embedded in $\mathbb{R}^3$ we define the Weingarten map as

$$\mathcal{W}_{p,S} = (I)^{-1}(II)$$

where I is the matrix of the first fundamental form and II that of the second.

$$I = \begin{pmatrix} E & F \\ F & G \end{pmatrix},$$
$$II = \begin{pmatrix} L & M \\ M & N \end{pmatrix}. \qquad \square$$

Principal Curvatures

We define the principal curvatures of a surface S at a point $p \in S$ to be the minimum and maximum of the normal curvatures of all curves passing through p. They can be found using the following:

For principal curvatures k_1, k_2 of S at p, and tangent vectors $t_1, t_2 \in T_pS$ corresponding to the principal curvatures we have

$$\mathcal{W}_p(t_i) = k_i t_i.$$

In other words, the principal curvatures are the eigenvalues of the Weingarten map. Furthermore, we call the eigenvectors t_1, t_2 the principal directions of S at p. $\square$

Gaussian Curvature

We can compute the Gaussian curvature K in several ways, one of which is

$$K = k_1 k_2.$$

Another one that is useful for computation is

$$K = Det(\mathcal{W}_{p,S}),$$

or equivalently

$$K = \frac{Det(II)}{Det(I)}.$$

Mean Curvature

Similar to the Gaussian curvature we can find the mean curvature in several ways:

$$H = \frac{k_1 + k_2}{2},$$

another one being

$$H = \frac{1}{2}\text{Trace}(\mathcal{W}_{p,S}),$$

or equivalently

$$H = \frac{LG - 2MF + NE}{2Det(I)}.$$

Higher Dimensional

If S is a manifold embedded in $\mathbb{R}^{n+1}$, then the Weingarten map is an n dimensional square matrix, and we can use the following definitions:

$$K = \prod_{i=1}^{n} k_i,$$
$$H = \frac{1}{n}\sum_{i=1}^{n} k_i. \qquad \square$$

Weingarten Surfaces

Mean-Gauss surfaces as defined by (1) are a particular case of Weingarten Surfaces. A class of surfaces whose mean and Gaussian curvatures are connected by a function f in the sense that[1]

$$f(H,K) = 0. \tag{2}$$

For Mean-Gauss surfaces we have:

$$\begin{aligned}
\frac{k_1 + k_2}{2} &= \lambda' k_1 k_2, \\
k_1 &= \lambda k_1 k_2 - k_2, \\
k_1 - \lambda k_1 k_2 &= -k_2, \\
k_1(1 - \lambda k_2) &= -k_2, \\
k_1 &= \frac{-k_2}{(1 - \lambda k_2)}, \\
k_1 &= \frac{k_2}{k_2\lambda - 1}.
\end{aligned}$$

So we define f to be

$$f(H,K) = W(k_1,k_2) = k_1 - \frac{k_2}{k_2\lambda - 1}. \tag{3}$$

The set of solutions to (2) are called the curvature diagram, or W-diagram of a surface.

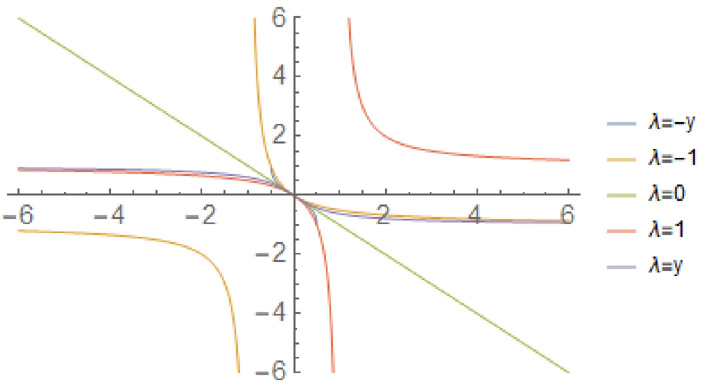

Figure 1. W-diagram of $[x - \frac{y}{y\lambda-1}]$ (2) for various λ.

We have Theorem 1 giving relations concerning said diagrams:

Theorem 1 (Interpretation of Principal Curvatures[1]). *If the curvature diagram of a surface S*

i) *degenerates to exactly one point, then S has two constant principal curvatures and is part of a plane, sphere, or circular cylinder.*
ii) *is contained in one of the coordinate axes through the origin, then S is developable.*
iii) *is contained in the main diagonal $k_1 = k_2$, then the surface S is part of a plane or sphere, as every point is umbilic.*

We also have the following bi-directional statements:

i) *The curvature diagram is contained in a straight line parallel to the diagonal $k_1 = -k_2$ if and only if the mean curvature is constant.*
ii) *The curvature diagram is contained in a standard hyperbola $k_1 = \frac{c}{k_2}$ for $c \in \mathbb{R}$ if and only if the Gaussian curvature is constant.* □

Spheres and Hyperspheres

One family of surfaces for which the Mean-Gauss relation is satisfied are spheres and hyperspheres.

Consider the sphere $S = \{(x,y,z)|x^2+y^2+z^2 = \rho^2\}$, which has the parametric equation

$$S(\rho,\theta,\phi) = (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)$$

where $\theta \in [0,2\pi]$ is the azimuthal angle (longitude), $\phi \in [0,\pi]$ is the polar angle (co-latitude), and ρ the radius of S.

Then S has the following Gaussian and mean curvatures.

$$K = \frac{1}{\rho^2},$$
$$H = \frac{1}{\rho}.$$

An important remark is that both quantities are constant based on the radius of our sphere.

Now, by the above we see that the unit sphere satisfies the condition with $\lambda = 1$, and for $\lambda \neq 1$ we have

$$\begin{aligned} H_S &= \lambda K_S, \\ \frac{1}{\rho} &= \lambda\frac{1}{\rho^2}, \\ \lambda &= \rho. \end{aligned}$$

For hyperspheres of radius ρ, we take a different approach by using the Gauss and Weingarten map.

Recall that the Gauss map $\mathcal{G} : S \to \mathbb{S}^n$, takes our surface to the unit ball for $\mathbb{R}^n$. But here our surface is already a sphere, so we simply scale it.

Note that we are taking the geometer's approach; using n to denote the dimension of the sphere.

So if $S = \{x \in \mathbb{R}^{n+1} | \sum_{i=1}^{n+1} x_i^2 = \rho^2\}$ is a sphere of radius ρ embedded in R^{n+1}, then for $x \in S$, the unit normals and by extension the Gauss map will be

$$\mathcal{G}(S) = -(\frac{S}{\rho})$$

which maps every $x \in S$ to $-\frac{x}{\rho}$, giving us that for every $x \in S$ we have

$$\begin{aligned} \sum_{i=1}^{n+1}(-\frac{x_i}{\rho})^2 &= \frac{1}{\rho^2}\sum_{i=1}^{n+1} x_i^2 \\ &= \frac{1}{\rho^2}\rho^2 \\ &= 1, \end{aligned}$$

which describes the unit sphere $\mathbb{S}^n$. Remark that we chose the negative normal direction; the reasoning becomes apparent in the next step.

Now,

$$\mathcal{W} = -D\mathcal{G},$$

the negative of the Jacobian of the Gauss map, which is given by

$$\mathcal{W} = (\frac{1}{\rho})\mathcal{I}$$

where $\mathcal{I}$ is the identity matrix for $\mathbb{R}^n$, since $D : T_p\mathbb{S}^n \to T_p\mathbb{S}^n$.

The curvatures are then given by the trace of $\mathcal{W}$ over n and its determinant for the other.

$$H_S = \frac{1}{n}\text{Trace}(\mathcal{W}) = \frac{1}{\rho}$$
$$K_S = Det(\mathcal{W}) = \frac{1}{\rho^n}$$

We see that the Mean-Gauss relation is satisfied for $\lambda = \rho^{n-1}$.

Thus, we have proven Theorem 2 as follows:

Theorem 2 (n-Sphere). *For any n-sphere S of radius ρ, the Mean-Gauss relation*

$$H_S = \lambda K_S$$

is satisfied by $\lambda = \rho^{n-1}$. ■

(a) Gaussian Curvature

(b) Mean Curvature

Figure 2. Sphere coloured according to curvature.

If we compare this result with the standard case of $n = 2$, we can see that it is consistent.

Trivial Cases

It should be mentioned that there are trivial cases of this problem. First, if the surface S is a plane which has 0 Gaussian and mean curvature, then $H = \lambda K$ is trivially satisfied.

Another case which can be considered trivial or rather one we can immediately rule out is that of non-planar developable surfaces. In this case, one of the principal curvatures will be 0, giving

$$H = \lambda(0)$$

which lacks a solution for λ in the non-extended reals.

The last trivial case to be considered is that of minimal surfaces with $H_S = 0$ satisfied by $\lambda = 0$. Therefore, we restrict $\lambda \in \mathbb{R}\backslash\{0\}$.

Graphs of Functions

Let $S \subset \mathbb{R}^3$ be a graph; that is, $S = \{(x, y, z) \in \mathbb{R}^3 | f(x, y) = z\}$ for some function f which is continuous and well defined.

Then, the coefficients of the first fundamental form of S are given by

$$\begin{aligned} E &= ||(1,0,f_x)||^2 & F &= (1,0,f_x)(0,1,f_y) & G &= ||(0,1,f_y)||^2 \\ &= 1 + f_x^2, & &= f_x f_y, & &= 1 + f_y^2. \end{aligned}$$

And we have a unit normal to S,

$$\begin{aligned} \vec{n} &= \frac{(1,0,f_x) \times (0,1,f_y)}{||(1,0,f_x) \times (0,1,f_y)||} \\ &= \frac{(-f_x, -f_y, 1)}{\sqrt{1 + f_x^2 + f_y^2}} \\ &= \frac{(-f_x, -f_y, 1)}{d}, \end{aligned}$$

where $d = \sqrt{1 + f_x^2 + f_y^2}$. Now, the coefficients of the second fundamental form are

$$\begin{aligned} L &= (0,0,f_{xx}) \cdot \vec{n} & M &= (0,0,f_{xy}) \cdot \vec{n} & N &= (0,0,f_{yy}) \cdot \vec{n} \\ &= \frac{f_{xx}}{d}, & &= \frac{f_{xy}}{d}, & &= \frac{f_{yy}}{d}. \end{aligned}$$

Using the definition of the Weingarten map

$$\mathcal{W}_{p,S} = (I)^{-1}(II),$$

we have

$$\mathcal{W}_{p,S} = \frac{1}{EG - F^2} \begin{pmatrix} GL - FM & GM - FN \\ ME - FL & NE - FM \end{pmatrix},$$

yielding the curvatures

$$H_S = \frac{1}{2} \frac{GL - 2FM + NE}{EG - F^2}$$

(a) Gaussian Curvature

(b) Mean Curvature

Figure 3. Graph of $f(x, y) = \sin(x)\cos(y)$ coloured according to curvature.

$$
\begin{aligned}
&= \frac{1}{2}\frac{\frac{(1+f_y^2)(f_{xx})}{d} - 2\frac{f_xf_yf_{xy}}{d} + \frac{(1+f_x^2)(f_{yy})}{d}}{(1+f_x^2)(1+f_y^2) - f_x^2f_y^2} \\
&= \frac{1}{2}\frac{1}{d}\frac{(1+f_y^2)(f_{xx}) - 2f_xf_yf_{xy} + (1+f_x^2)(f_{yy})}{1+f_x^2+f_y^2} \\
&= \frac{1}{2d^3}[(1+f_y^2)(f_{xx}) - 2f_xf_yf_{xy} + (1+f_x^2)(f_{yy})],
\end{aligned}
$$

and

$$
\begin{aligned}
K_S &= \frac{LN-M^2}{EG-F^2} \\
&= \frac{\frac{f_{xx}f_{yy}}{d^2} - \frac{f_{xy}^2}{d^2}}{d^2} \\
&= \frac{f_{xx}f_{yy} - f_{xy}^2}{d^4}.
\end{aligned}
$$

Now we want to meet the relation $H_S = \lambda K_S$ for λ a non-zero constant. As such, we can leave out the $\frac{1}{2}$ scalar from the mean curvature, giving us

$$
\begin{aligned}
\frac{1}{d^3}[(1+f_y^2)(f_{xx}) - 2f_xf_yf_{xy} + (1+f_x^2)(f_{yy})] &= \lambda\frac{f_{xx}f_{yy} - f_{xy}^2}{d^4}, \\
d[(1+f_y^2)(f_{xx}) - 2f_xf_yf_{xy} + (1+f_x^2)(f_{yy})] &= \lambda[f_{xx}f_{yy} - f_{xy}^2]
\end{aligned}
$$

Immediately, we note a trivial solution to this equation which we covered in the *Trivial Cases* section: flat surfaces where the second order partials are 0, giving $0 = 0$.

So assume $f_{xx}f_{yy} - f_{xy}^2 \neq 0$ which rules out said trivial case, giving

$$
d\frac{[(1+f_y^2)(f_{xx}) + (1+f_x^2)(f_{yy})]}{[f_{xx}f_{yy} - f_{xy}^2]} = \lambda.
$$

Then for λ to be a constant, either both d and $\frac{[(1+f_y^2)(f_{xx}) + (1+f_x^2)(f_{yy})]}{[f_{xx}f_{yy} - f_{xy}^2]}$ are constants, or they are reciprocals of each other as functions.

Case I: Let d be a constant; that is, $\sqrt{1+f_x^2+f_y^2} = c$ for some $c \in \mathbb{R}$. Then clearly, f_x^2 and f_y^2 must be constants or $f_x^2 + f_y^2 = \hat{c}$, a constant.

If both are constant, then f is a plane which is the trivial case and not being considered here.

If not both constant, then $f_x^2 + f_y^2 = \hat{c}$, for some $\hat{c} \in \mathbb{R}$. Then f can be of the forms

$$
f_- = c_1 + yc_2 - x\sqrt{\hat{c} - c_2^2}
$$

and

$$
f_+ = c_1 + yc_2 + x\sqrt{\hat{c} - c_2^2}
$$

for constant c_1 and c_2. But then $f_{xx} = f_{yy} = f_{xy} = 0 \Rightarrow$ we have a contradiction to $f_{xx}f_{yy} - f_{xy}^2 \neq 0$. Note that this is equivalent to the second fundamental form being a 0 matrix.

Therefore, d is not a constant, and we are in the case that they are reciprocals of each other.

Case II: We have

$$
\begin{aligned}
\frac{[(1+f_y^2)(f_{xx}) + (1+f_x^2)(f_{yy})]}{[f_{xx}f_{yy} - f_{xy}^2]} &= \frac{\lambda}{d}, \\
\frac{GdL + EdN}{dLdN - d^2M^2} &= \frac{\lambda}{d}, \\
\frac{d}{d^2}\frac{GL+EN}{LN-M^2} &= \frac{\lambda}{d}, \\
\frac{1}{d}\frac{GL+EN}{LN-M^2} &= \frac{\lambda}{d}, \\
\frac{GL+EN}{LN-M^2} &= \lambda.
\end{aligned}
$$

To continue, we will need a few definitions and propositions:

Definition 2 (Umbilical Points[2 (p178)]). *We say that a point p is umbilical if*

$$
k_1(p) = k_2(p)
$$

for principal curvatures k_1, k_2. □

Proposition 1 (Spheres are totally umbilic). *Every point p on a sphere S is umbilic.*

The proof of which is immediate from a calculation which yields $k_{1,2} = \frac{1}{\rho}$ where ρ is the radius of the sphere. Furthermore, spheres are the only totally umbilic surfaces with non-zero curvatures. □

Proposition 2 (Diagonal Fundamental Forms[2 (p201)]). *Let p be a point of a surface S, and suppose that p is not an umbilic. Then, there is a surface patch $\sigma(u,v)$ of S containing p whose first and second fundamental forms are*

$$
Edu^2 + Gdv^2 \quad \text{and} \quad Ldu^2 + Ndv^2,
$$

respectively, for some smooth functions E, G, L, N. □

Now assume that S is not totally umbilic; then, there exists a point p of S for which locally the equation in case II becomes

$$
\frac{GL+EN}{LN} = \frac{G}{N} + \frac{E}{L} = \lambda. \tag{4}
$$

Furthermore, by the same proposition we have that

$$
\begin{aligned}
F &= 0, \\
f_xf_y &= 0
\end{aligned}
$$

which implies either $f_x = 0$ or $f_y = 0$ in a neighborhood of p. Without loss of generality, let $f_y = 0$, then $N = \frac{f_{yy}}{d} = 0$, and from (4),

$$
\frac{G}{N} + \frac{E}{L} = \frac{1}{0} + \frac{E}{L}
$$

is undefined.

Therefore, for S satisfying the mean Gauss relation there are no such points p, and we have that S is totally umbilic. Since the sphere is the only surface with non-zero curvatures and which is totally umbilic, we have that S must be part of a sphere.

Thus we have proven the following Theorem 3:

Theorem 3 (Mean-Gauss Graphs). *Let $S = \{(x,y,z) \in \mathbb{R}^3 | z = f(x,y)\}$, for a function f which is smooth and well defined, be the graph of a function such that $H_S = \lambda K_S$ for $\lambda \in \mathbb{R}\backslash\{0\}$ and such that $f_{xx}f_{yy} - f_{xy}^2 \neq 0$. Then, S is a part of or a whole sphere.*

We note that for S to be the graph of a function and a whole sphere, we will require more than one chart. ■

Surfaces of Revolution

Let $C = (x(s), y(s))$ be a regular smooth curve parameterized by arc length, which generates a surface of revolution S in the following way:

$$S = \{(x(s), y(s)\cos(t), y(s)\sin(t))\}.$$

We call C the profile curve of S. For brevity going forward, we will omit s when writing such that $S = \{(x, y\cos(t), y\sin(t)\}$.

We have first order partial derivatives

$$S_s = (x', y'\cos(t), y'\sin(t)), \qquad S_t = (0, -y\sin(t)), y\cos(t)),$$

and second

$$\begin{aligned} S_{ss} &= (x'', y''\cos(t), y''\sin(t)), \qquad S_{tt} = (0, -y\cos(t)), -y\sin(t)),\\ S_{st} &= (0, -y'\sin(t), y'\cos(t)). \end{aligned}$$

Then the first fundamental form of S has the components

$$\begin{aligned} E &= ||S_s||^2 \\ &= x'^2 + y'^2\cos(t)^2 + y'^2\sin(t)^2 \\ &= x'^2 + y'^2(\cos^2(t) + \sin^2(t)) \\ &= x'^2 + y'^2 \\ &= ||C||^2 \\ &= 1, \\ F &= -y'y\sin(t)\cos(t) + yy'\sin(t)\cos(t) \\ &= 0, \end{aligned} \qquad \begin{aligned} G &= ||S_t||^2 \\ &= y^2(\sin^2(t) + \cos^2(t)) \\ &= y^2. \end{aligned}$$

In matrix form,

$$\mathcal{I} = \begin{pmatrix} 1 & 0 \\ 0 & y^2 \end{pmatrix}.$$

In order to compute the second fundamental form, we start with the unit normal:

$$\begin{aligned} \vec{n} &= \frac{S_s \times S_t}{||S_s \times ||S_t} \\ &= \frac{\begin{bmatrix} x' & y'\cos(t) & y'\sin(t) \\ 0 & -y\sin(t) & y\cos(t) \end{bmatrix}}{d} \\ &= \frac{[(y'y\cos^2(t) + y'y\sin^2(t)), -(x'y\cos(t)), (-x'y\sin(t))]}{d} \\ &= \frac{[y'y, -x'y\cos(t), -x'y\sin(t)]}{[y'^2y^2 + x'^2y^2\cos^2(t) + x'^2y^2\sin^2(t)]^{\frac{1}{2}}} \\ &= \frac{[y'y, -x'y\cos(t), -x'y\sin(t)]}{y[y'^2 + x'^2(\cos^2(t) + \sin^2(t))]^{\frac{1}{2}}} \\ &= \frac{[y'y, -x'y\cos(t), -x'y\sin(t)]}{y} \\ &= (y', -x'\cos(t), -x'\sin(t)). \end{aligned}$$

Now the components of the second fundamental form are:

$$\begin{aligned} L &= S_{ss} \cdot \vec{n} \\ &= (x'', y''\cos(t), y''\sin(t))(y', -x'\cos(t), -x'\sin(t)) \\ &= x''y' - y''x'\cos^2(t) - y''x'\sin^2(t) \\ &= x''y' - y''x', \\ M &= s_{st} \cdot \vec{n} \\ &= (0, -y'\sin(t), y'\cos(t))(y', -x'\cos(t), -x'\sin(t)) \\ &= x'y'\sin(t)\cos(t) - x'y'\sin(t)\cos(t) \\ &= 0, \\ N &= S_{tt} \cdot \vec{n} \\ &= (0, -y\cos(t)), -y\sin(t))(y', -x'\cos(t), -x'\sin(t)) \\ &= yx'\cos^2(t) + yx'\sin^2(t) \\ &= yx', \end{aligned}$$

giving the matrix form

$$\mathcal{II} = \begin{pmatrix} x''y' - y''x' & 0 \\ 0 & yx' \end{pmatrix},$$

with mean and Gaussian curvature

$$\begin{aligned} H_S &= \frac{1}{2}\frac{GL - 2FM + NE}{EG - F^2} \\ &= \frac{1}{2}\frac{y^2(x''y' - y''x') + yx'}{y^2} \\ &= \frac{1}{2}\frac{y(x''y' - y''x') + x'}{y}, \\ K_S &= \frac{LN - M^2}{EG - F^2} \\ &= \frac{(x''y' - y''x')yx'}{y^2} \\ &= \frac{(x''y' - y''x')x'}{y}. \end{aligned}$$

Now putting it in the form of the relation (1), we get:

$$\begin{aligned} H_S &= \lambda' K_S, \\ \frac{y(x''y' - y''x') + x'}{y} &= \lambda\frac{(x''y' - y''x')x'}{y}, \\ y(x''y' - y''x') + x' &= \lambda(x''y' - y''x')x', \\ y(x''y' - y''x') + x' - \lambda(x''y' - y''x')x' &= 0, \\ (x''y' - y''x')(y - \lambda x') + x' &= 0. \end{aligned}$$

Case I: $(x''y' - y''x') = 0$

Then $x' = 0 \Rightarrow x = c$, for some constant c, and either S degenerates to a single point or is a $yz-$plane passing through $x = c$.

Case II: $(y - \lambda x') = 0$

Then $x' = 0 \Rightarrow y = 0$ and S is given by $S = \{(c, 0, 0)\}$ a single point.

Case III: $(x''y' - y''x')(y - \lambda x') = -x'$

For case III, we take a different approach. Let $(x''y' - y''x')$, $(y - \lambda x')$, and x' be non-zero and recall we parameterized C by arc length, so we have $x'^2 + y'^2 = 1$ giving several relations. For the moment we are interested in

$$\begin{aligned} x'^2 + y'^2 = 1 \Rightarrow 2x'x'' + 2y'y'' &= 0, \\ x'' &= \frac{-y'y''}{x'}. \end{aligned}$$

Then from above, we recalculate the mean and Gaussian curvatures

$$
\begin{aligned}
H_S &= \frac{1}{2}\frac{y(x''y' - y''x') + x'}{y} \\
&= \frac{1}{2}\frac{y([\frac{-y'y''}{x'}]y' - y''x') + x'}{y} \\
&= \frac{1}{2}\left(\frac{-y'^2y'' - y''x'^2}{x'} + \frac{x'}{y}\right) \\
&= \frac{1}{2}\left(\frac{-y''}{x'}(y'^2 + x'^2) - \frac{x'}{y}\right) \\
&= \frac{1}{2}\left(\frac{x'}{y} - \frac{y''}{x'}\right),
\end{aligned}
$$

$$
\begin{aligned}
K_S &= \frac{(x''y' - y''x')x'}{y} \\
&= \frac{x'([\frac{-y'y''}{x'}]y' - y''x')}{y} \\
&= \frac{-y'^2y'' - y''x'^2}{y} \\
&= \frac{-y''(y'^2 + x'^2)}{y} \\
&= \frac{-y''}{y},
\end{aligned}
$$

giving the new relation

$$
\begin{aligned}
H &= \hat{\lambda}K, \\
\frac{1}{2}\left(\frac{x'}{y} - \frac{y''}{x'}\right) &= \hat{\lambda}\frac{-y''}{y}, \\
\left(\frac{x'}{y} - \frac{y''}{x'}\right) &= \lambda\frac{y''}{y}, \qquad \lambda = -2\hat{\lambda}.
\end{aligned}
$$

Now we once again refer to the parameterization by arc length which gives that for $x, y \in [0,1]$ we have the expression $x' = \pm\sqrt{1-y'^2}$. We keep in mind the symmetry of surfaces of revolution and consider only the positive case:

$$
\begin{aligned}
\left(\frac{\sqrt{1-y'^2}}{y} - \frac{y''}{\sqrt{1-y'^2}}\right) &= \lambda\frac{y''}{y}, \\
\left(\frac{1-y'^2-yy''}{y\sqrt{1-y'^2}}\right)\left(\frac{y}{y''}\right) &= \lambda, \\
\left(\frac{1-y'^2-yy''}{y''\sqrt{1-y'^2}}\right) &= \lambda, \\
(1-y'^2-yy'') &= \lambda(y''\sqrt{1-y'^2}).
\end{aligned}
$$

From here, recall that our original profile curve was given by $(x(s), y(s))$, a function of s. Make the substitution $z = y' = \frac{dy}{ds}$. This gives

$$
\begin{aligned}
y'' &= \frac{dz}{ds} = \frac{dz}{dy}\cdot\frac{dy}{ds} \\
&= z\frac{dz}{dy}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
(1-z^2-yz\frac{dz}{dy}) &= \lambda(z\frac{dz}{dy}\sqrt{1-z^2}), \\
1-z^2 &= \lambda z\frac{dz}{dy}\sqrt{1-z^2} + yz\frac{dz}{dy}), \\
1-z^2 &= \frac{dz}{dy}\left(\lambda z\sqrt{1-z^2} + yz\right), \\
\frac{dy}{dz} &= \frac{\lambda z}{\sqrt{1-z^2}} + \frac{yz}{(1-z^2)}. \qquad (*)
\end{aligned}
$$

Equation $(*)$ is a non-homogeneous linear differential equation in y, so we begin by solving the homogeneous case:

$$
\begin{aligned}
\frac{dy}{dz} &= \frac{yz}{(1-z^2)}, \\
\frac{1}{y}dy &= \frac{z}{(1-z^2)}dz, \\
\ln(y) &= \frac{-1}{2}\ln(1-z^2) + C, \\
\ln(y) &= \ln(\frac{1}{\sqrt{1-z^2}}) + C, \\
e^{\ln(y)} &= e^{\ln(\frac{1}{\sqrt{1-z^2}})+C}, \\
e^{\ln(y)} &= e^{\ln(\frac{1}{\sqrt{1-z^2}})}e^C, \\
y &= \frac{D(z)}{\sqrt{1-z^2}}. \qquad (**)
\end{aligned}
$$

Then differentiating with respect to z gives us:

$$
\begin{aligned}
y' &= \frac{D'[\sqrt{1-z^2}] - D[\frac{-z}{\sqrt{1-z^2}}]}{(1-z^2)} \\
&= \frac{D'}{\sqrt{1-z^2}} + \frac{Dz}{(1-z^2)^{\frac{3}{2}}} \\
&= \frac{D'}{\sqrt{1-z^2}} + \frac{yz}{(1-z^2)}.
\end{aligned}
$$

Returning to $(*)$, the non-homogeneous case, by plugging in our found homogeneous solution, we obtain

$$
\begin{aligned}
\frac{\lambda z}{\sqrt{1-z^2}} + \frac{yz}{(1-z^2)} &= \frac{D'}{\sqrt{1-z^2}} + \frac{yz}{(1-z^2)}, \\
\lambda z &= D'.
\end{aligned}
$$

Therefore, D' is a linear function giving

$$
D = \lambda\frac{z^2}{2} + R
$$

where R is a constant. Then from $(**)$ and using $\lambda = -2\hat{\lambda}$,

$$
\begin{aligned}
y &= \frac{-\hat{\lambda}z^2 + R}{\sqrt{1-z^2}} \\
&= \frac{R - \hat{\lambda}z^2}{\sqrt{1-z^2}}.
\end{aligned}
$$

Case IIIa: If $\hat{\lambda} \neq R$, a computer algebra system[3] (the process of which can be seen in reference) gives the results:

$$
c_1 - \frac{s}{\sqrt{2}} = \int_1^{y(s)} \frac{1}{\sqrt{\frac{-\sqrt{\xi^2\left(-4ab+4b^2+\xi^2\right)}+2ab-\xi^2}{b^2}}}d\xi,
$$

$$
c_1 + \frac{s}{\sqrt{2}} = \int_1^{y(s)} \frac{1}{\sqrt{\frac{-\sqrt{\zeta^2\left(-4ab+4b^2+\zeta^2\right)}+2ab-\zeta^2}{b^2}}}d\zeta,
$$

$$c_1 - \frac{s}{\sqrt{2}} = \int_1^{y(s)} \frac{1}{\sqrt{\frac{\sqrt{\vartheta^2\left(-4ab+4b^2+\vartheta^2\right)+2ab-\vartheta^2}}{b^2}}} d\vartheta,$$

$$c_1 + \frac{s}{\sqrt{2}} = \int_1^{y(s)} \frac{1}{\sqrt{\frac{\sqrt{\varphi^2\left(-4ab+4b^2+\varphi^2\right)+2ab-\varphi^2}}{b^2}}} d\varphi.$$

At first glance, the linearity of the left hand side leads to a contradiction of the assumption that we are in a non-trivial case; however, further research into the implications of the solution are required.

For the this paper we will focus on the more immediate case:

Case IIIb: Now if $\hat{\lambda} = R$, we get:

$$\begin{aligned}
y &= \hat{\lambda}\frac{1-z^2}{\sqrt{1-z^2}} \\
&= \hat{\lambda}\sqrt{1-z^2}, \\
y^2 &= \hat{\lambda}^2(1-z^2) \\
&= \hat{\lambda}^2 - \hat{\lambda}^2 z^2, \\
y^2 - \hat{\lambda}^2 &= -\hat{\lambda}^2 z^2, \\
\frac{\hat{\lambda}^2 - y^2}{\hat{\lambda}^2} &= z^2, \\
\frac{\sqrt{\hat{\lambda}^2 - y^2}}{\hat{\lambda}} &= z, \\
\frac{1}{\hat{\lambda}} &= \frac{z}{\sqrt{\hat{\lambda}^2 - y^2}}.
\end{aligned}$$

Then, by definition we have $z = y'$, giving:

$$\begin{aligned}
\int \frac{1}{\hat{\lambda}} ds &= \int \frac{y'}{\hat{\lambda}\sqrt{1-(\frac{y}{\hat{\lambda}})^2}} ds, \\
\frac{s}{\hat{\lambda}} + \phi &= \arcsin(\frac{y}{\hat{\lambda}}).
\end{aligned}$$

Note that we used the substitution $u = \frac{y}{\hat{\lambda}}$ to integrate. Finally, we get an expression for the component function $y(s)$:

$$y(s) = \hat{\lambda}\sin(\frac{s}{\hat{\lambda}} + \phi).$$

We remark here that the constant ϕ represents a phase shift on the input angle of our function. Now from the arc-length parameterization:

$$\begin{aligned}
x'^2 + y'^2 &= 1, \\
x'^2 + (\hat{\lambda}\cos(\frac{s}{\hat{\lambda}} + \phi)\frac{1}{\hat{\lambda}})^2 &= 1, \\
x'^2 &= 1 - (\cos(\frac{s}{\hat{\lambda}} + \phi))^2, \\
x' &= \pm\sin(\frac{s}{\hat{\lambda}} + \phi), \\
x(s) &= \mp\hat{\lambda}\cos(\frac{s}{\hat{\lambda}} + \phi) + \gamma.
\end{aligned}$$

So $x(s)$ takes the form of a general sinusoidal function; we have reflections along the x-axis given by $\mp$, amplitude control via $\hat{\lambda}$, phase shifts from ϕ, and finally translations along x handled by γ.

So the profile curve C of S meeting the criteria is given by

$$C = (\mp\hat{\lambda}\cos(\frac{s}{\hat{\lambda}} + \phi) + \gamma, \hat{\lambda}\sin(\frac{s}{\hat{\lambda}} + \phi)),$$

which is a circle centered on the x axis at $x = \gamma$ of radius $\hat{\lambda}$. Then the surface of revolution S is

$$S = \{\mp\hat{\lambda}\cos(\frac{s}{\hat{\lambda}} + \phi), \hat{\lambda}\sin(\frac{s}{\hat{\lambda}} + \phi)\cos(t), \hat{\lambda}\sin(\frac{s}{\hat{\lambda}} + \phi)\sin(t))\},$$

which is a sphere of radius $\hat{\lambda}$, centered at $(\gamma, 0, 0)$.

A fascinating result is that depending on our definition of the domain of s, namely if $|\mathrm{Dom}(s)| < |2\hat{\lambda}\pi|$ where $|\cdot|$ represents the standard Lebesgue measure, we obtain part of a sphere which still meets the criteria. So we can extend our hypothesis to include:

If $H = \lambda K$ for a surface S as above and non-zero λ, then S is part of a sphere.

Which gives a solid basis for the following conjecture:

Conjecture 1 (Mean-Gauss Surfaces of Revolution)**.** *Given $C = (x(s), y(s))$, a complete smooth curve parameterized by arc length, which generates a surface of revolution S in the following way:*

$$S = \{(x(s), y(s)\cos(t), y(s)\sin(t))\}$$

for which we have $H_S = \lambda K_S$ for $\lambda \in \mathbb{R}\backslash\{0\}$ and $K_S \neq 0$ globally.

Then S is a sphere, or part of a sphere with profile curve:

$$\begin{aligned}
x(s) &= \mp\hat{\lambda}\cos(\frac{s}{\hat{\lambda}} + \phi) + \gamma, \\
y(s) &= \hat{\lambda}\sin(\frac{s}{\hat{\lambda}} + \phi).
\end{aligned} \qquad \square$$

An important remark is that if we were instead to chose to rotate about the y axis, we would obtain a similar result but having instead:

$$\begin{aligned}
y(s) &= \mp\hat{\lambda}\cos(\frac{s}{\hat{\lambda}} + \phi) + \gamma, \\
x(s) &= \hat{\lambda}\sin(\frac{s}{\hat{\lambda}} + \phi).
\end{aligned}$$

And since the phase shift can be chosen as required to fit our initial conditions we can put the equations in standard form for $\mathbb{R}^2$ namely:

$$\begin{aligned}
y(s) &= \mp\hat{\lambda}\sin(\frac{s}{\hat{\lambda}} + \phi') + \gamma, \\
x(s) &= \hat{\lambda}\cos(\frac{s}{\hat{\lambda}} + \phi').
\end{aligned}$$

Surfaces of Constant Curvature

For surfaces of constant curvature, there are three possible cases. One and two are relatively uninteresting in our context, but they will be briefly covered.

i) Constant mean curvature surfaces
ii) Constant Gaussian curvature surfaces
iii) Both mean and Gaussian curvatures are constant

Case I: Constant Mean Curvature

If $H = 0$, then we have the trivial case of a plane or a minimal surface

(a) Gaussian Curvature

(b) Mean Curvature

Figure 4. Enneper's Minimal Surface coloured according to curvature.

which is satisfied by $\lambda = 0$. As previously mentioned, it is excluded from the general case of our relation.

And if $H = \gamma$ for some $\gamma \in \mathbb{R}$, but $K = f(x)$ is non constant for $x \in \mathbb{R}^3$, then there is clearly no λ constant which will satisfy

$$\gamma = \lambda f(x)$$

for every x on the surface. ■

Case II: Constant Gaussian Curvature

In this case, if $K = 0$ then our relation is not satisfied for any $\lambda \in \mathbb{R}$, as discussed in *Trivial Cases*.

And as in Case I, if $H = f(x)$ and $K = \gamma$ then

$$f(x) = \lambda\gamma$$

has no solutions which hold for every x on the surface. ■

Case III: Constant Mean and Gaussian Curvature

To handle case III, we need a theorem and its corollary from Montiel and Ros[4].

Theorem 4 (Classification of Surfaces with Parallel Second Fundamental Form)**.** *An orientable surface whose principal curvatures are constant, or equivalently, whose Gauss and mean curvatures are constant, is necessarily an open subset of a plane, a sphere, or of a right circular cylinder.* □

Corollary 1. *The only connected surfaces closed as subsets of $\mathbb{R}^3$ having constant principal curvatures are planes, sphere and right circular cylinders.* □

From the above corollary, we need only consider the 3 surfaces. If we have a right circular cylinder we are in the case that $K = 0$ and the relation is not satisfied. The plane is once again the trivial case, leaving us with the sphere which we have shown satisfies our relation.

Thus we have proven the following Theorem 5:

Theorem 5 (Mean-Gauss Surface of Constant Curvature)**.** *The only orientable surfaces of constant curvature (mean or Gaussian) which satisfies $H = \lambda K$ non-trivially are spheres.* ■

Mean-Gauss Theorems

Collected Relation Theorems

For ease of reference, find the collected theorems from the previous sections below.

Spheres

For any n-sphere S of radius ρ, the Mean-Gauss relation

$$H_S = \lambda K_S$$

is satisfied by $\lambda = \rho^{n-1}$. ■

Graphs

Let $S = \{(x, y, z) \in \mathbb{R}^3 | z = f(x, y)\}$, for a function f which is smooth and well defined, be the graph of a function such that $H_S = \lambda K_S$ for $\lambda \in \mathbb{R}\backslash\{0\}$ and such that $f_{xx}f_{yy} - f_{xy}^2 \neq 0$. Then S is a part of or a whole

(a) Gaussian Curvature

(b) Mean Curvature

Figure 5. Pseudo-Sphere has $K_S = -1$ globally.

sphere. ■

Surfaces of Revolution (Conjecture)

Given $C = (x(s), y(s))$, a complete smooth curve parameterized by arc length, which generates a surface of revolution S in the following way:

$$S = \{(x(s), y(s)\cos(t), y(s)\sin(t))\}$$

for which we have $H_S = \lambda K_S$ for $\lambda \in \mathbb{R}\backslash\{0\}$ and $K_S \neq 0$ globally.

Then

$$\begin{aligned} x(s) &= \mp\hat{\lambda}\cos(\frac{s}{\hat{\lambda}} + \phi) + \gamma, \\ y(s) &= \hat{\lambda}\sin(\frac{s}{\hat{\lambda}} + \phi), \end{aligned}$$

and S is a sphere, or part of a sphere. □

Surfaces of Constant Curvature

The only orientable surfaces of constant curvature (mean or Gaussian) which satisfies $H = \lambda K$ non-trivially are spheres. ■

We also state a version of the Implicit function theorem from Lang[5] which refers to it as "The Implicit Mapping Theorem." Note that some of the statements of the theorem in our version have been made more specific as Serge Lang is dealing with general Banach spaces.

Theorem 6 (Implicit Function Theorem)**.** *Let $f : \mathbb{R}^{m+n} \to \mathbb{R}^m$ where $(x, y) \mapsto f(x, y)$ be a continuously differentiable function. Then for a fixed point $(a, b) \in \mathbb{R}^{m+n}$ with $f(a, b) = 0$, if the Jacobian matrix*

$$J_{f,y}(a,b) = \begin{bmatrix} \frac{\partial f_1}{\partial y_1}(a,b) & \dots & \frac{\partial f_1}{\partial y_m}(a,b) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial y_1}(a,b) & \dots & \frac{\partial f_m}{\partial y_n}(a,b) \end{bmatrix}$$

is invertible, then there exists an open set $U \subset \mathbb{R}^n$ containing (a) such that there exists a unique continuously differentiable function $g : U \to \mathbb{R}^m$ for which $g(a) = b$ and $f(x, g(x)) = 0$ for all $x \in U$. □

In essence, the implicit function theorem tells us that for any surface S, there exist points p_i for which in a neighborhood of the image of p_i we can describe the surfaces as the graph of a function.

Theorem 7 (Weak Mean-Gauss Theorem)**.** *The only embedded smooth non-developable surfaces $S \subset \mathbb{R}^3$ containing at least one point (a, b) satisfying the condition $J_{f,y}(a, b)$ is invertible, and which globally satisfies $H_S = \lambda K_S$ where $\lambda \in \mathbb{R}$ are part of or a complete sphere.*

Proof. The proof by contradiction is immediate from the relation for graphs of functions (Theorem 3) and the implicit function theorem.

Let S be a surface as described and not part of a sphere, then locally there would be a patch for which the surface writes as the graph of function which is not locally spherical, hence it fails the relation. ■

Conjecture 2 (Strong Mean-Gauss Conjecture)**.** *The only smooth non-developable surfaces $S \subset \mathbb{R}^3$ satisfying the Mean-Gauss relation $H_S = \lambda K_S$ for $\lambda \in \mathbb{R}$ are part of or a whole sphere.* □

While the proven theorems in this paper point towards the strong conjecture holding, more research is needed in order to prove it concretely, and even further work needs to be done in order to prove an n dimensional version of the theorems.

Another interesting question is that if we relax the global requirement for smoothness, what types of surfaces can we build which satisfy the relation?

One simple example is if we have two spheres of equal radii, we can take two parts and glue them together. Then the relation is satisfied everywhere but along the glued boundary where the classical notion of curvature fails to exist.

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