Dynamics of the financial market

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Financial mathematics must make use of assumptions in the development of mathematical models that provide predictive power on the behavior of economic markets, as it is impossible to collect data on the market as a whole. As a result, important quantities, such as the risk-measurement of a portfolio, are often inaccurately estimated. The financial market seems to be an erratic, pattern-less system. Indeed, attempts to find patterns, and to explain the processes behind the price movements of an asset, have been largely unsuccessful. This is analogous to the 'Turkey Problem' described by N. Taleb in his book "The Black Swan". To illustrate, a turkey spends its life being fed and raised for slaughter, a fact that is unbeknownst to it. From the point of view of the turkey, life is delicious and predictable, until the day it is killed. For the turkey, its death is a 'black swan event', as it represents something highly unpredictable and catastrophic. This same type of uncertainty is also present in financial markets.

Numerous examples of events with a significant impact on the system demonstrate the failure of the current model can be found. The fate of Long Term Capital Management (LTCM) is one such example. LTCM failed to predict the collapse of the Russian ruble, as it assumed that the Russian government defaulting on its bonds would be a highly unlikely event. Current theory describes these catastrophic events as so highly improbable that their very occurrence is near miraculous, and cannot be predicted. This would be a reasonable theory if there was only one such event, perhaps two, in the course of history. However, one can find plenty of examples of these so-called impossible events by examining historical data. This raises the question; if these events are happening with a higher frequency than expected, is our current model correct? It is important to remember that markets do not obey a model, but rather the model seeks to explain the behavior of markets. As such, we must attempt to construct a model that resembles and predicts our observed data, or the model should be deemed invalid.

Central to the classic financial model is the Gaussian or normal distribution. If a sample of numbers is normally distributed it should cluster around a mean or average value, and the likelihood of an event is increasingly rare as it deviates further from this mean. The occurrence and magnitude of these deviations are described by the variance of the distribution. As such, the normal distribution is described by two natural parameters, the mean and the variance. The normal distribution is followed by many natural systems, and as a result this mathematical model has found applications in many diverse fields from astronomy to population dynamics.

When describing a natural phenomenon, such as the stock market, we seek to produce a model that is able to match an observed pattern. For example, in financial time series, we consider the difference between the natural logarithm of the prices. To illustrate this, consider a time series

$$(P_0, P_1, \ldots, P_{t-1}, P_t, P_{t+1}, \ldots)$$

where the subscript t indicates time intervals. Consider the set of values

$$X(t) = \log(P_{t+1}) - \log(P_t)$$

This is our variable of interest, the unit-free difference in stock prices. It is this variable which classical financial theory assumes to be normally distributed. In other words, we expect the value of a price change over one time interval to be close in value to the mean of the normal distribution, with significant deviations away from the mean being quite rare. If for example we assume our changes to follow a standard normal distribution, which means it has a mean of zero and a variance of one, we would expect the price difference to be zero, so our prices would be constant, and large price fluctuations (whether positive or negative) to be rare. The other important condition that is imposed on this model is that each event is independent. Assuming the price changes follow a standard normal distribution, we can base models on Brownian motion, which is the same concept that describes the movements of particles in fluid. This concept is the foundation for the derivation of the famous Black-Scholes equation, which is used to anticipate market movements by generating a probable value for equity pricing.

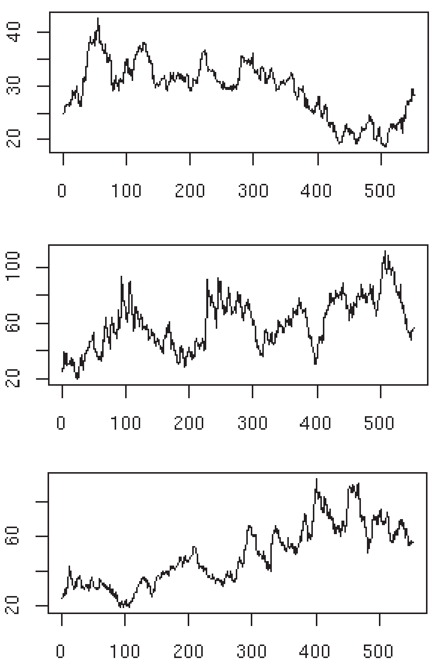
Now consider our variable of interest, the change in stock prices, and examine the consequences of assuming this variable to be drawn from the Gaussian curve. Our first assumption is to expect the data to be centered around a mean, which has been shown to be true empirically (Tsay, 2005). However, a problem arises in discussing "outliers" in the data, points that are at least two standard deviations away from the mean. In financial data, one is often presented with these anomalies. For example, consider the stock market crash in 1987, or the internet boom in the 1990s. These events deviated widely from the trend. We can also consider the magnitude of these outlying events. Yet based on the Gaussian distribution model, the odds of a value falling far away from the mean is fairly low. Can the Gaussian model be applied to financial data? Does the normal distribution allow for absolute price movements ten times the average return?

We will consider the weekly closing prices of General Electric as listed by Yahoo! Finance from January 8, 1962 to September 4, 2007. General Electric is an ideal choice to represent the 'average' asset on the market, as its ample data, size and market diversity reflects general market trends. This provides us with 2382 observations which will allows us to use large sample properties. If we assume that the price changes are normally distributed we can find the estimated parameters of the normal distribution. In the case of GE we find the mean = $0.0002487107 and the variance = $0.002518528.

Within this data set I have identified eight statistically impossible events. These events represent large price changes. While one such event, such as the 1987 stock crash, may be accepted as a statistical anomaly, the occurrence of eight such events over a time period of 45 years is statistically improbable and contradicts the prediction of our model. We therefore suspect that the returns are not normally distributed. The very existence of these outliers shows that there are observations that cannot be explained by the current theory. It seems that these events occur frequently and generally have lasting impact on the markets. Shouldn't a model that claims to understand the dynamics of price changes take these events into account?

The second essential assumption made is that price movements are independent of each other. Intuitively, one would think that this is not true at all. In fact, consider an individual purchasing stock. The purchase is made based on the stock's past performance, particularly its recent performance. By standardizing our sample (simply subtracting the mean and dividing by the variance) we can assess whether these price movements are independent by analyzing increasing or decreasing streaks in the data. If price changes are independent, there should not be any noticeable streaks in the data (large consecutive positive or negative price movements). With this in mind, we find that indeed our data reflect the presence of increasing and decreasing streaks, which are improbable under the classic Gaussian model.

An alternative theory has been explored. Benoit Mandelbrot has developed the area of fractal finance in order to explain these issues that other models have dismissed. He demonstrates that this fractal view of the market, which in some ways is actually an elegant generalization of the Brownian motion concept, fits the observed data more closely than the Gaussian model (Mandelbrot 2004). Simply put, fractals are mathematical objects that look the same when viewed at both low and high resolution. In the case of stock prices, increasing the resolution means looking at smaller time scales. To understand this, consider the following three graphs of



GE stock returns. It is impossible to distinguish between the graphs with time scales of days, weeks and months.

This fractal property coupled with a random element seems to be a better model for price changes. The Multifractal Model of Asset Returns (MMAR) was introduced by Mandelbrot. X(t) is a compound process, such that

$$X(t) = B_H[\theta(t)]$$

Where $B_H(t)$ is a fractional Brownian motion operator with self-affinity index H ,and $\theta(t)$ is the stochastic trading time. While the components of this model reflect complex mathematics, we can still understand how it works.

The self-affinity index H accounts for the observation that no matter what time scale you consider, the system looks the same. This allows the model to compensate for streaks in the data, something which our classical model fails to do. For example, large values of H at some given time result in persistent trading (trading in the same direction), while low values of H indicate very little movement. The stochastic trade time $\theta(t)$, is called a multi-fractal process. This captures the fluctuations in the observed volatility of the data. In other words, this allows the model to take into account what has happened in the past, while allowing for extreme price changes, extremely improbable in the classical model. In refining a model in this fashion, a particular question arises; is our new model more general than the previous model? Alternatively, can we, under certain restrictive conditions, obtain our classical model from our fractal model? As we cannot derive the classical model from the fractal model, if the fractal model is indeed legitimate, the classical model cannot be right, and thus has no real value in describing financial markets.

To conclude, we have explored the current view of the financial markets, assuming that a simple Brownian engine drives the observed price changes in the financial market. We utilized real data and tested the assumptions of this model to see whether the actual data followed the proposed model. Specifically, we identified a set of events occurring relatively frequently which would be impossible based on the classical model. We also noticed the presence of trends in the price changes which provide evidence against the independence of these returns. This in turn implied that our model should incorporate some sort of memory of the past, so we proceeded to incorporate the property of time scale invariance. With this in mind, we presented the MMAR model, which reflected these observations. Under the current mode of thought, models like the Black-Scholes pricing equation underestimate the potential of large increases or decreases in value. These are not merely potential events, but have already occurred, including the collapse of the Russian ruble, the collapse of the subprime mortgage market or Long Term Capital Management, among other financial disasters in history. While we cannot be certain that markets obey the MMAR model, at the very least we can affirm that it is a more accurate representation of financial returns, rendering the classical model obsolete.

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